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BOUNDEDNESS AND CONTINUITY OF THE FUNDAMENTAL SOLUTION OF THE TIME DEPENDENT SCHRÖDINGER EQUATION WITH SINGULAR POTENTIALS

Dedicated to Professor Kyûya Masuda on his sixtieth birthday

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Abstract. We show that the fundamental solution of the initial value problem for the time dependent Schrödinger equation is bounded and continuous for a class of non-smooth potentials. The class is large enough to accomodate Coulomb potentials if the spatial dimension is three.

1. Introduction. We consider the Cauchy problem for the time dependent Schrödinger equation

$$(1.1)\qquad i\frac{\partial u}{\partial t}=-\frac{1}{2}\Delta u+V(x)u\,,\quad (t,x)\in \boldsymbol{R}^1\times\boldsymbol{R}^m\,;\quad u(0,x)=u_0(x)\,,\quad x\in\boldsymbol{R}^m$$

in the Hilbert space $L^2(\boldsymbol{R}^m)$. We assume that the potential $V(x)$ is real-valued and the operator $-(1/2)\Delta+V$ on $C_0^\infty(\boldsymbol{R}^m)$ defines a unique selfadjoint extension H in $L^2(\boldsymbol{R}^m)$. Then, the equation (1.1) has a unique solution $u(t)=e^{-itH}u_0$. The distribution kernel $E(t,x,y)$ of the propagator e^{-itH} is called the fundamental solution (FDS for short) of (1.1):

$$u(t,x)=e^{-itH}u_0(x)=\int E(t,x,y)u_0(y)dy\,.$$

The FDS $E(t,x,y)$ is a solution of (1.1) with the initial data $E(0,x,y)=\delta(x-y)$. In this paper, we show that $E(t,x,y)$ is continuous and bounded, $|E(t,x,y)|\le C_T|t|^{-m/2}$ for $0<|t|\le T<\infty$, for a class of potentials $V(x)$ which can be as singular as $|x|^{-(m-\varepsilon)/(m-1)}$ and decay at infinity as slowly as $V(x)=o(1)$. The class is wide enough to accommodate Coulomb potentials $V(x)=\sum_{j=1}^N Z_j/|x-R_j|$ in dimension three.

When $V(x)$ is C^∞, it was recently shown that the smoothness property of the FDS is determined mainly by the growth rate of $V(x)$ at infinity: The FDS is smooth and bounded for $t\neq 0$ if V is subquadratic, viz., $|V(x)|=o(|x|^2)$ roughly speaking ([20], see also [21], [11], [3]); whereas $E(t,x,y)$ is nowhere C^1 if V is superquadratic in dimension one, viz., $V(x)\ge C|x|^{2+\varepsilon}$, $\varepsilon>0$ near infinity ([20]); and at the borderline case $|V(x)|\sim$

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