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HYPERGEOMETRIC POLYNOMIALS OVER FINITE FIELDS

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Abstract. Honda found certain hypergeometric polynomials over the prime field which can be expressed as a product of linear factors. In this paper we give a different proof of his result by using elementary functions described by hypergeometric series of Gauss. We can find another hypergeometric polynomials which Honda missed.

Introduction. Let p be any odd prime and let $\boldsymbol{F}_p$ denote the prime field of characteristic p. For any a in $\boldsymbol{F}_p$ and $0 \leq n \in \boldsymbol{Z}$, we put $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ if $n \geq 1$. For any a, b, c in $\boldsymbol{F}_p$, we define the hypergeometric polynomials over the finite field $\boldsymbol{F}_p$ by

$$\boldsymbol{F}(a, b, c; x) = \sum_{n=0} \frac{(a)_n (b)_n}{(1)_n (c)_n} x^n ,$$

where we stop the sum as soon as the numerator vanishes and assume that the denominator does not vanish before the numerator does.

These polynomials have already appeared in Deuring [1], Igusa [4], Ihara [5], Honda [3] and others.

Honda [3] proved the following result: Let $p \geq 5$ and let ϕ denote the Legendre character of $\boldsymbol{F}_p$. For any $\varepsilon, \varepsilon' \in \{1, -1\}$, we define

$$S_{\varepsilon,\varepsilon'} = \{a \in \boldsymbol{F}_p^\times \mid \phi(a) = \varepsilon,\ \phi(1-a) = \varepsilon'\} ,$$

$$F_{\varepsilon,\varepsilon'}(x) = \prod_{a \in S_{\varepsilon,\varepsilon'}} (x-a) .$$

Then Honda proved:

THEOREM 0.1 (Honda).

$$F_{-1,-1}(x) = a^{(p)}_{-1,-1} \cdot \boldsymbol{F}\left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2}; x\right),$$

$$F_{-1,1}(x) = a^{(p)}_{-1,1} \cdot \boldsymbol{F}\left(\frac{1}{4}, \frac{3}{4}, \frac{3}{2}; x\right),$$

where $a^{(p)}_{\varepsilon,\varepsilon'}$ *is a constant which takes value* 1, 2 *or* 1/2.

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