ON THE DEFINITION OF CESÀRO-PERRON INTEGRALS

YÔTO KUBOTA

(Received February 11, 1959)

1. Introduction. The Cesàro-Perron integral was defined by J. C. Burkill [1]*) using the Cesàro-continuous upper and lower functions.

G. Sunouchi and M. Utagawa [3] proved that the Cesàro-Perron scale of integration can be defined without assuming the Cesàro-continuity of upper and lower functions and that the indefinite integral is Cesàro-continuous.

We denote by CP_0 and CP the Burkill's Cesàro-Perron integral and the generalized Cesàro-Perron integral defined by G. Sunouchi and M. Utagawa respectively. It is clear that CP-integral includes CP_0-integral. But, in this paper, we will prove the equivalence of these integrals by using the Cesàro-Denjoy integral introduced by W. L. C. Sargent [2].

I must express my best thanks to Dr. G. Sunouchi for his suggestions and criticisms.

2. CP_0-integral and CP-integral.

DEFINITION 2. 1. We put

$$C(f, a, b) = \frac{1}{b-a}\int_a^b f(t)\, dt,$$

where the integral is taken in the restricted Denjoy sense.

If $\lim\limits_{h\to 0} C(f, x_0, x_0 + h) = f(x_0)$, then $f(x)$ is termed *Cesàro-continuous at* x_0.

If $\overline{CD}\, f(x_0) = \underline{CD}\, f(x_0)$, where

$$\overline{\lim_{h\to 0}} \left\{C(f, x_0, x_0 + h) - f(x_0)\right\} \Big/ \frac{1}{2}h = \overline{CD}\, f(x_0)$$

and

$$\underline{\lim_{h\to 0}} \left\{C(f, x_0, x_0 + h) - f(x_0)\right\} \Big/ \frac{1}{2}h = \underline{CD}\, f(x_0),$$

then $f(x)$ is called *Cesàro differentiable at* x_0 and we denote the common value by $CD\, f(x_0)$.

DEFINITION 2. 2. $U(x)$ $[L(x)]$ is termed *upper* [*lower*] *function of a measurable* $f(x)$ *in* $[a, b]$, provided that

*) Numbers in brackets refer to the bibliography at the end.