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OPERATORS WITH α-CLOSED RANGE

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Let $\mathscr{H}$ denote a Hilbert space of infinite dimension h. In an earlier work we have introduced the notion of α-closed subspace, where α is a cardinal, $\aleph_0 \leqq \alpha \leqq h$ (definition 2.1 of [2]). A subspace $\mathscr{K}$ of $\mathscr{H}$ is called α-closed if there is a closed subspace $\mathscr{L}$ of $\mathscr{H}$ such that $\mathscr{L} \subset \mathscr{K}$ and such that

$$\dim(\mathscr{L}^{\perp} \cap \mathscr{K})^{-} < \alpha .$$

This notion is of interest only when $\alpha > \aleph_0$, since a subspace $\mathscr{K}$ is $\aleph_0$-closed if and only if it is closed (lemma 2.3 of [2]). This concept is important for the study of operators on nonseparable spaces, as it is used in characterizing invertibility modulo the closed two-sided ideals of the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$. (Cf. definition 2.7 and theorems 2.6 and 2.8 of [2].) For each α, $\aleph_0 \leqq \alpha \leqq h$, let $\mathscr{I}_\alpha$ denote the set of operators of rank $\rho(A)$ less than α and let $\mathscr{J}_\alpha$ denote the norm closure of $\mathscr{I}_\alpha$. Then the $\mathscr{J}_\alpha$, $\aleph_0 \leqq \alpha \leqq h$ are precisely the closed two-sided ideals of $\mathscr{L}(\mathscr{H})$, and the elements of $\mathscr{J}_\alpha$ are called α-compact operators. (Cf. [5] and theorem 0 of [2].) In this terminology, the $\aleph_0$-compact operators are precisely the compact operators. Then the operators which are invertible modulo $\mathscr{J}_\alpha$ (i.e., the operators A in $\mathscr{L}(\mathscr{H})$ for which there exists an operator A' in $\mathscr{L}(\mathscr{H})$ such that $I - AA' \in \mathscr{J}_\alpha$ and $I - A'A \in \mathscr{J}_\alpha$) are precisely the α-Fredholm operators. An operator A is called α-Fredholm if its range is α-closed and its nullity $\nu(A) < \alpha$ and its corank $\rho'(A) < \alpha$. In this context we see that the notion of an operator having α-closed range is fundamental for the study of operators on a nonseparable space. We therefore examine this new concept in more detail in this paper.

We first obtain a characterization of operators with α-closed range as those which have closed range modulo the ideal $\mathscr{I}_\alpha$. More explicitly, A has α-closed range if and only if there exists an operator C in $\mathscr{I}_\alpha$ such that $A + C$ has closed range. This enables us to generalize the well-known fact that A has closed range if and only if A^* has closed range, to the case of operators with α-closed range.

We give two applications of this result. The first gives the conditions