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BESOV SPACES AND SOBOLEV SPACES ON A NILPOTENT LIE GROUP

KŌICHI SAKA

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Introduction. In this paper we shall study the theory of Besov spaces (or Lipschitz spaces) and Sobolev spaces on a nilpotent Lie group. To admit a wide variety of applications to more problems we consider the class of "stratified groups" as a class of nilpotent Lie groups. On such Lie groups there is a natural notion of homogeneity which enables one to duplicate many of the standard constructions of Euclidean spaces. But we can not yet duplicate most of results in the theory of Fourier transforms and distributions. Hence fractional integral operators play a fundamental role in our paper. These operators have been extensively studied by G. B. Folland [5], A. Yoshikawa [24] and H. Komatsu [10], [11], [12], [13],]14], [15] in a general setting. By employing the Bessel potential as one of these fractional integral operators we develop the theory of Besov spaces and Sobolev spaces on a stratified group. Our paper is heavily influenced by Flett's paper [4].

The plan of our paper is as follows: In Section 1 we present notations used in later sections and recall the necessary background material concerning homogeneous structures on nilpotent Lie groups. In Section 2, we consider the diffusion semigroup generated by the sub-Laplacian $\mathfrak{J}$ on a stratified group, and we use it to define the Bessel potentials given as fractional powers of the operator $(1+\mathfrak{J})$. Further, we discuss properties of the semigroup, its kernel function and the Bessel potentials. In Section 3 we define an analogue of the classical Besov space in terms of the Bessel potentials and extend several basic theorems from the Euclidean case to our case. Further we investigate several equivalent spaces to this Besov space. In Section 4 we shall see that this Besov space coincides with that defined by use of the Poisson semigroup for positive fractional powers. In Section 5 we define an analogue of the classical Sobolev space in terms of the Bessel potentials. We see that this space has an alternative representation in terms of "the Riesz potentials" and we use it to prove the inclusion theorem in this section and the interpolation theorem in the next section. Several basic theorems for the interpolation space of Besov spaces and Sobolev spaces are dis-