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ORBITS AND THEIR ACCUMULATION POINTS OF CYCLIC SUBGROUPS OF MODULAR GROUPS

Dedicated to Professor Tatuo Fuji'i'e on his sixtieth birthday

HARUMI TANIGAWA

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Introduction. As is well known, the Teichmüller space $T(S)$ of a Riemann surface S of finite analytic type (p, n) with $3p-3+n>0$ is a complex manifold of dimension $3p-3+n$, and is complete with respect to the Teichmüller metric. Bers [B2] gave a classification of modular transformations in terms of the translation lengths, and showed that the types of modular transformations are characterized by the self-mappings of S inducing them. By definition and facts shown in [B2], hyperbolic modular transformations are expected to have properties similar to that of hyperbolic Möbius transformations. For example Bers [B2] showed that a non-periodic modular transformation is hyperbolic if and only if it has an invariant Teichmüller line, and gave a remark (without proof) that for each hyperbolic modular transformation the invariant line is unique by Thurston's theory. He also posed a problem to prove the uniqueness of the invariant line using quasiconformal mappings. In this paper we show (§2) this by combining the theory of quasiconformal mappings and the result of Bowen and Marcus [BM]. Using this fact we give a simple proof of the theorem of McCarthy about the centralizer and normalizer of a hyperbolic cyclic subgroup of the modular group (Theorem 2.4).

It is also well-known that the Teichmüller space $T(S)$ is identified with a bounded domain of $\boldsymbol{C}^{3p-3+n}$, via the embedding introduced by Bers, and each point of the boundary corresponds to a Kleinian group. From the discontinuity of the action of the modular group, for every non-periodic modular transformation $[f]_*$, induced by a self-mapping $f: S\to S$, and for a point $\tau\in T(S)$ the accumulation points set of the sequence $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ is contained in the boundary of $T(S)$. Interesting investigations about relations between the type of the modular transformation $[f]_*$ and the type of the Kleinian groups corresponding to accumulating points of $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ are seen in [B3], [S], and [H]. It is natural to expect that the Kleinian group corresponding to an accumulation point of the sequence $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ inherits the property of f, if the mapping f has some symmetric property. This line of thought is developed in §3. The argument there yields a different way of approach to necessary conditions, studied by Birman, Lubotsky and McCarthy [BLM], for two non-hyperbolic modular transformations to commute.

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