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THE STRUCTURE OF THE POLYTOPE ALGEBRA

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Abstract. We construct an isomorphism from McMullen's polytope algebra, onto the quotient of the algebra of continuous, piecewise polynomial functions with integral value at 0, by its ideal generated by coordinate functions. This explains the non-trivial grading of the polytope algebra, by the obvious grading of piecewise polynomial functions. In the process of the proof, we make explicit many connections between convex polytopes and piecewise polynomials.

Introduction. In the study of valuations (or finitely additive measures) on convex polytopes in a finite-dimensional real vector space, a fundamental role is played by the polytope algebra: the universal group for translation-invariant valuations. This group is endowed with a multiplication, via Minkowski sum of polytopes, and with many other structures, discovered by McMullen, Morelli, Khovanskii-Pukhlikov and others. In particular, the polytope algebra is almost a graded algebra over $\boldsymbol{R}$; its grading is defined by diagonalizing the action of the group of dilatations (see [Mc1]). The proof of existence of this grading uses the logarithm of a polytope P, defined by $\log(P)=\sum_{n=1}^{\infty}(-1)^{n-1}(P-1)^n/n$ (this makes sense in the polytope algebra, because $P-1$ is nilpotent there).

In this paper, we recover some of the most important properties of the polytope algebra, as corollaries of a structure theorem for this algebra. To state our main result, we need some notation.

Let V be a vector space over $\boldsymbol{R}$ of finite dimension $d\geq 2$, and let V^* be its dual. To any convex polytope P in V^* is associated its support function H_P on V; then H_P is continuous, and piecewise linear with respect to some subdivision of V into polyhedral cones having the origin as their common vertex. We denote by R the algebra of all continuous functions on V that are piecewise polynomial (in the same sense). Then R is a graded algebra over $\boldsymbol{R}$ for the operations of pointwise addition and multiplication; it turns out that R is generated by support functions of polytopes. We denote by $\bar{R}$ the quotient of R by its graded ideal generated by all (globally) linear functions on V.

THEOREM. (i) *The graded algebra* $\bar{R}=\bigoplus_{n=0}^{\infty}\bar{R}_n$ *vanishes in all degrees* $n>d$. *Moreover, the vector space* $\bar{R}_d$ *is one-dimensional, and multiplication in* $\bar{R}$ *induces non-degenerate pairings* $\bar{R}_j\times\bar{R}_{d-j}\to\bar{R}_d$ *for* $1\leq j\leq d-1$.

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