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DISTINCT ITERABLE BRANCHES

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§1. Introduction. The basic problem of inner model theory is how to construct mice satisfying hypotheses appreciably stronger than "there is a Woodin limit of Woodin cardinals". We have a family of constructions, the K^c*-constructions*, which ought to produce such mice under the appropriate hypotheses on V. Perhaps the most important thing we lack is a proof that the countable elementary submodels of premice produced by a K^c-construction are $\omega_1 + 1$-iterable. The best partial results in this direction are those of Neeman ([4]) for K^c-constructions making use of full background extenders over V, and those of Andretta, Neeman, and Steel ([1]) for arbitrary K^c-constructions.

Let $\mathscr{M}$ be a countable premouse embedded by π into a level of the K^c-construction $\mathbb{C}$. If $\mathbb{C}$ uses only full extenders over V as its background extenders, then π and $\mathbb{C}$ enable one to lift an evolving iteration tree $\mathscr{T}$ on $\mathscr{M}$ to an iteration tree $\mathscr{T}^*$ on V. (See [3, §12].) The good behavior of $\mathscr{T}^*$ guarantees that of $\mathscr{T}$. The natural conjecture here is that V is $\omega_1 + 1$-iterable *with respect to such trees* $\mathscr{T}^*$ by the strategy of choosing the unique wellfounded branch. The open question here is uniqueness, since by [2] the uniqueness of the wellfounded branch chosen by $\mathscr{T}^*$ at limit stages strictly less than λ implies the existence of a wellfounded branch to be chosen at λ.[1] There is reasonably good evidence for the truth of this conjecture, at least in the case π is lexicographically minimal among all embeddings of $\mathscr{M}$ to a model of $\mathbb{C}$. (See [5] for a discussion.)[2]

In many situations, one must rely on K^c-constructions making use of partial background extenders. Given a countable $\mathscr{M}$ embedded by π into a level of such a construction $\mathbb{C}$, it is no longer possible to use π and $\mathbb{C}$ to lift iteration trees on $\mathscr{M}$ to trees on V, and so the conjecture of the last paragraph doesn't make sense. Instead of branches which lift to wellfounded branches of a tree on V, what we get here from the basic existence Theorem of [6, §9], for countable trees $\mathscr{T}$ on $\mathscr{M}$, is a maximal branch b of $\mathscr{T}$ such that b is π-*realizable*. (If b does not drop, then letting $\mathscr{N}$ be the target model for π, this means that there is a $\sigma\colon \mathscr{M}_b^{\mathscr{T}} \to \mathscr{N}$ such that $\pi = \sigma \circ i_b^{\mathscr{T}}$. See [6] for the full definition.) There may be more than one π-realizable branch (see [2, §5]), but if π is lexicographically minimal with respect

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[1] Here we assume the background extenders of $\mathbb{C}$ are all $2^{\aleph_0}$-closed. This can be arranged easily in applications.

[2] Woodin has constructed an iteration tree on V having distinct cofinal, wellfounded branches, however, his tree is not induced by a tree on a premouse.

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