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NORMAL SUBGROUPS OF INFINITE SYMMETRIC GROUPS, WITH AN APPLICATION TO STRATIFIED SET THEORY

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It is generally known that infinite symmetric groups have few nontrivial normal subgroups (typically only the subgroups of bounded support) and none of small index. (We will explain later exactly what we mean by *small*). However the standard analysis relies heavily on the axiom of choice. By dint of a lot of combinatorics we have been able to dispense—largely—with the axiom of choice. Largely, but not entirely: our result is that if X is an infinite set with $|X| = |X \times X|$ then $\mathrm{Symm}(X)$ has no nontrivial normal subgroups of small index. Some condition like this is needed because of the work of Sam Tarzi who showed [4] that, for any finite group G, there is a model of ZF without AC in which there is a set X with $\mathrm{Symm}(X)/\mathrm{FSymm}(X)$ isomorphic to G.

The proof proceeds in two stages. We consider a particularly useful class of permutations, which we call the class of *flexible* permutations. A permutation of X is flexible if it fixes at least $|X|$-many points. First we show that every normal subgroup of $\mathrm{Symm}(X)$ (of small index) must contain every flexible permutation. This will be theorem 4. Then we show (theorem 7) that the flexible permutations generate $\mathrm{Symm}(X)$.

However the study of infinite symmetric groups without AC—gripping tho' it is—was not our point of departure. The reason why we wished to prove this result was that the universes of Russell-like typed set theories (and of Quine's NF) satisfy the the condition above: $|V| = |V \times V|$, and if the symmetric group for such a universe has no nontrivial normal subgroups of small index then the theory of sets definable in such universes can be shown to be very simple. We need to eschew AC because AC fails in NF. Readers with doubts about Con(NF) should not on those grounds dismiss these constructions since they work also in NFU+Infinity, a system whose consistency is secure. We devote the second part of the paper to an exploration of these very satisfying consequences. See Jensen: [3].

§1. The group theoretic result. Throughout this section we shall suppose that X is an infinite set with $|X| = |X \times X|$. Since products are only defined up to isomorphism, we may take $X \times X = X$, with projection maps p_1 and p_2. With distinct a and b in X, we have an injection $\mathscr{P}(X) \to \mathrm{Symm}(X)$; $Y \mapsto \{(\langle y, a\rangle, \langle y, b\rangle) : y \in Y\}$. But we also have an injection $\mathrm{Symm}(X) \to \mathscr{P}(X \times X) \to \mathscr{P}(X)$. So $|\mathrm{Symm}(X)| = |\mathscr{P}(X|)$. Similarly, we obtain $|X \sqcup X| = |X|$ and $|X| + 1 = |X|$. This

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