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BOREL COMPLEXITY OF ISOMORPHISM BETWEEN QUOTIENT BOOLEAN ALGEBRAS

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§1. Introduction and nomenclature.

1.1. History of the question. In response to a question of Farah, "How many Boolean algebras $\mathscr{P}(\mathbb{N})/\mathcal{I}$ are there?" [Far04], one of us (Oliver) proved that there are continuum-many nonisomorphic Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ with $\mathcal{I}$ a Borel ideal on the natural numbers, and in fact that this result could be improved simultaneously in two directions:

(i) "Borel ideal" may be improved to "analytic P-ideal"
(ii) "continuum-many" may be improved to "E_0-many"; that is, E_0 is Borel reducible to the isomorphism relation on quotients by analytic P-ideals.

See [Oli04].

In [AdKech00], Adams and Kechris showed that the relation of equality on Borel sets (and therefore, any Borel equivalence relation whatsoever) is Borel reducible to the equivalence relation of Borel bireducibility. (In somewhat finer terms, they showed that the partial order of inclusion on Borel sets is Borel reducible to the quasi-order of Borel reducibility.) Their technique was to find a collection of, in some sense, strongly mutually ergodic equivalence relations, indexed by reals, and then assign to each Borel set B a sort of "direct sum" of the equivalence relations corresponding to the reals in B. Then if $B_1 \subseteq B_2$ it was easy to see that the equivalence relation thus induced by B_1 was Borel reducible to the one induced by B_2, whereas in the opposite case, taking x to be some element of $B_1 \setminus B_2$, it was possible to show that the equivalence relation corresponding to x, which was part of the equivalence relation induced by B_1, was not Borel reducible to the equivalence relation corresponding to B_2.

The purpose of the current work is to show that every Borel equivalence relation is reducible to the isomorphism relation on quotients by Borel ideals, and we shall follow approximately the same general plan that was used by Adams and Kechris. However there are a couple of significant differences.

First, note that B will in general be uncountable, so the "direct sum" is over uncountably many objects. For Adams and Kechris this was not a problem; they could consider a Polish space in "two dimensions", letting $\langle x_0, x_1\rangle$ be equivalent to

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