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SIMPLICIAL STRUCTURES IN MV-ALGEBRAS AND LOGIC

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§1. Introduction. Classical logic, as is well known, can be analyzed in a great part by algebraic methods using the Lindenbaum algebra obtained from the formal system. For example the completeness theorem for this logic becomes equivalent to the semisimplicity of the obtained Lindenbaum algebra.

Since Chang [4, 5], Łukasiewicz logic has also been analyzed algebraically through the associated Lindenbaum type algebra, that is the algebra of equivalence classes obtained from the relation of provable equivalence. In this case this algebra is an MV-algebra [4]. Once again logical notions have an algebraic counterpart, for example, completeness relates strongly to semisimplicity [4, 5]. However, unlike the classical case where the algebras in question are Boolean and always semisimple, not all MV-algebras are semisimple. This fact, in a sense, enriches the theory of MV-algebras.

Now every MV-algebra can be considered a Lindenbaum type algebra, namely an algebra associated to Łukasiewicz logic with additional axioms. Thus we can carry over to any MV-algebra various logical notions such as (in)completeness, consistency, satisfiability, etc.

Two important logical notions are those of "formal consequence" and "semantical consequence". The former just says that a wff α is deducible from a set of wff $\mathscr{H}$ via the axioms and rules of inference, while the latter just says that every evaluation that "satisfies" all the members of $\mathscr{H}$ also "satisfies" α.

Informally call these relations $\mathbf{F}$, $\mathbf{S}$ respectively; consider them as binary relations, $\mathscr{H}\mathbf{F}\alpha$ and $\mathscr{H}\mathbf{S}\alpha$.

Now the completeness theorem just states $\mathbf{F} = \mathbf{S}$. Thus we can talk about an MV-algebra being "complete" provided the associated relations $\mathbf{F}$, $\mathbf{S}$ are equal.

In the case where the associated relations are not equal we shall compare them by using simplicial complexes. This provides a kind of "measure of the completeness" of a given MV-algebra.

Following Dowker [7], we attach to $\mathbf{F}$, $\mathbf{S}$ respectively, simplicial complexes $K_{\mathbf{F}}$, $K_{\mathbf{S}}$ thereby enabling a comparison of $\mathbf{F}$, $\mathbf{S}$ by comparing the simplicial structure of $K_{\mathbf{F}}$ and $K_{\mathbf{S}}$. When $\mathbf{F} = \mathbf{S}$ we say the algebra is "logically complete"; when the simplices $K_{\mathbf{F}}$, $K_{\mathbf{S}}$ are the same we say the algebra is "simplicially complete", and when the homology groups are the same we say the algebra is "homologically complete". We begin a study of MV-algebras from this point of view. We provide, for a finite set

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