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GENERIC Σ^1_3 ABSOLUTENESS

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In this article we study the strength of Σ^1_3 absoluteness (with real parameters) in various types of generic extensions, correcting and improving some results from [3]. (In particular, see Theorem 3 below.) We shall also make some comments relating this work to the bounded forcing axioms BMM, BPFA and BSPFA.

The statement "Σ^1_3 absoluteness holds for ccc forcing" means that if a Σ^1_3 formula with real parameters has a solution in a ccc set-forcing extension of the universe V, then it already has a solution in V. The analogous definition applies when ccc is replaced by other set-forcing notions, or by class-forcing.

THEOREM 1. [1] *Σ^1_3 absoluteness for ccc has no strength; i.e., if ZFC is consistent then so is ZFC + Σ^1_3 absoluteness for ccc.*

The following results concerning (arbitrary) set-forcing and class-forcing can be found in [3].

THEOREM 2 (Feng-Magidor-Woodin). (a) *Σ^1_3 absoluteness for arbitrary set-forcing is equiconsistent with the existence of a* reflecting cardinal, *i.e., a regular cardinal κ such that $H(\kappa)$ is Σ_2-elementary in V.*
(b) *Σ^1_3 absoluteness for class-forcing is inconsistent.*

We consider next the following set-forcing notions, which lie strictly between ccc and arbitrary set-forcing: proper, semiproper, stationary-preserving and ω_1-preserving. We refer the reader to [8] for the definitions of these forcing notions.

Using a variant of an argument due to Goldstern-Shelah (see [6]), we show the following. This result corrects Theorem 2 of [3] (whose proof only shows that if Σ^1_3 absoluteness holds in a certain proper forcing extension, then in L either ω_1 is Mahlo or ω_2 is inaccessible).

THEOREM 3. *Σ^1_3 absoluteness for semiproper forcing has no strength.*

PROOF. By an ω_1-iteration P_0 of semiproper forcing with revised countable support, produce a generic G_0 such that $L[G_0]$ satisfies semiproper absoluteness for Σ^1_3 formulas with real parameters in L. This is possible as there are only ω_1 reals in L and semiproperness is preserved through iteration with revised countable support. We can assume that P_0 has cardinality ω_1 in $L[G_0]$, as if necessary we can follow P_0 by a Lévy collapse with countable conditions to ω_1. Thus we have $L[G_0] = L[X_0]$, where X_0 is a subset of ω_1.

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