TSUKUBA J. MATH
Vol. 7 No 2 (1983). 323—336

AUTOMORPHISMS OF CERTAIN ROOT LATTICES

By

Zenji KOBAYASHI and Jun MORITA

0. Introduction.

Let Δ be a reduced irreducible root system of type X_l in a Euclidean space V, in the sense of Bourbaki [1]. Then Δ generates a lattice Γ of rank l in V. We fix the lattice Γ. Let Δ' be another reduced irreducible root system in V, generating Γ, of type X_l. We investigated whether Δ' coincided with Δ, and found out that only the case of C_4 is exceptional. If X_l is not C_4 then Δ' is equal to Δ. This means that (V, Γ, X_l) determines Δ uniquely unless X_l is C_4. In case X_l is C_4, there are three root systems, generating Γ, of type C_4 in V. As we will explain afterward, these are verified by looking at the list of root systems in Bourbaki [1].

Let W be the Weyl group of Δ, and $O(\Gamma)$ the orthogonal group of Γ. Then $W \subseteqq O(\Gamma)$. Let D be the subgroup of $O(\Gamma)$ generated by all symmetries of the Dynkin diagram of Δ. Put $\widetilde{W}=\langle W, D\rangle$, the subgroup of $O(\Gamma)$ generated by W and D. Notice that $-I$ (minus identity) is contained in $\widetilde{W}$ (cf. [1], [5]). Then the fact in the previous paragraph can be described as follows. The group index $[O(\Gamma):\widetilde{W}]$ is 3 if $X_l=C_4$; 1 otherwise.

In this paper, we will calculate the index $[O(\Gamma):\widetilde{W}]$ in the case that Δ is the root system of a Kac-Moody Lie algebra of Euclidean type or of low rank hyperbolic type. Let A be a generalized Cartan matrix of Euclidean type or of hyperbolic type, and B the associated form. Let Δ, Γ and $O(\Gamma)$ be the root system of A, the root lattice of Δ and the orthogonal group of Γ associated with B, respectively. We denote by W (resp. D) the Weyl group (resp. the diagram automorphism group) of A. Put $\widetilde{W}=\langle W, D, -I\rangle$. It is known that the index $Ind(A)=[O(\Gamma):\widetilde{W}]$ is finite (cf. [1; Chap. 5, §4, Ex. 18], [11]). If A is symmetric, then we get $Ind(A)=1$ as a direct consequence of [7; Prop. 1.6] and [12; Theorem 2]. We will compute $Ind(A)$ explicitely when A is of Euclidean type, of rank 2 hyperbolic type or of rank 3 hyperbolic type. The most interesting case is when $A=\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. In this case, we will observe that a certain

Received April 19, 1983