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GENERATORS AND RELATIONS FOR COMPACT LIE ALGEBRAS

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1. Introduction.

The main purpose of this paper is to provide a system of generators and relations for each of the nine types of compact simple Lie algebras. Indeed, we are able to give a presentation of each such algebra which depends only on the finite Cartan matrix (A_{ij}) which is attached to the complexification of our compact algebra. One of the main results that lies behind our work is the Theorem of Serre which gives a presentation of the simple Lie algebras over the complex field attached to (A_{ij}).

Although our main interest is with the compact Lie algebras, we work in the generality of Kac-Moody Lie algebras (see [1], [4], [7], [8]). In this setting we will be able to provide a generalization of the compact simple Lie algebras. We realize these algebras as certain forms of the Kac-Moody algebra. More specifically, if (A_{ij}) is any indecomposable Cartan matrix which is non-Euclidean we let $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$) be the reduced (resp. standard) Kac-Moody Lie algebra over the complex field $\boldsymbol{C}$, (see Section 1 for more details). We define a real form $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$) of $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$), and show that $\mathcal{L}_C$ is the only simple homomorphic image of $\bar{\mathcal{L}}_C$. We then give generators and relations for $\bar{\mathcal{L}}_C$. The question of when $\bar{\mathcal{L}}_C=\mathcal{L}_C$ is equivalent to the question of when $\mathcal{L}_C=\bar{\mathcal{L}}_C$, and is a major unsolved question about Kac-Moody algebras. However, thanks to Serre's Theorem, we know $\mathcal{L}_C=\bar{\mathcal{L}}_C$, and hence $\mathcal{L}_C=\bar{\mathcal{L}}_C$, when (A_{ij}) is of finite type. This yields a presentation of $\mathcal{L}_C$ in this case.

The content of the paper is as follows. In Section 1 we recall the notation and a few facts about Kac-Moody algebras. In Section 2, the final section, we begin by making a study of the algebras $\mathcal{L}_C$ and $\bar{\mathcal{L}}_C$. We then go on to obtain a presentation of $\bar{\mathcal{L}}_C$, and then use this in dealing with the compact simple Lie algebras. We think it is interesting that analogues of the compact Lie algebras exist in the Kac-Moody setting. Moreover, just as the compact algebras are

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