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# REMARK ON SOME COMBINATORIAL CONSTRUCTION OF RELATIVE INVARIANTS

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It is a classical problem to determine the explicit form of relative invariants. However, if it is too complicated, it seems more important to know the mathematical structure of relative invariants than just to write down the all terms of them. Hence, in this paper, we suggest to use some principle to construct relative invariants in § 1, and as examples, we shall construct some relative invariants of $GL(n, \boldsymbol{C})$ on $\overset{3}{\wedge}\boldsymbol{C}^n$ for all $n\geqq 6$ (See Propositions 4.1, 4.3, and 4.5), including all relative invariants for $n=6, 7, 8, 9$. This work was done while the author was visiting Europe, and he would like to express his hearty thanks to Prof. H. Popp at Mannheim University in West Germany, and to Prof. D. Luna at Grenoble University in France for their mathematical stimulation and encouragement. The author also would like to express his hearty thanks to Prof. M. Sato who kindly explained his works for $n=6$.

**§ 1.** Let $\rho: G\rightarrow GL(V)$ be a finite-dimensional rational representation of a reductive algebraic group $G$, all defined over the complex number field $\boldsymbol{C}$.

A homogeneous polynomial $f(x)$ on $V$ is called a *relative invariant* if there exists a rational character $\chi: G\rightarrow \boldsymbol{C}^{\times}$ satisfying $f(\rho(g)x)=\chi(g)f(x)$ for all $g\in G$ and $x\in V$. Now let $S^{r}(V)$ be the all homogeneous polynomials of degree $r$ on $V$. Then the group $G$ acts on $S^{r}(V)$ as $(g\phi)(x)\overset{\text{def}}{=}\phi(\rho(g)^{-1}x)$ for $\phi\in S^{r}(V)$, $g\in G$ and $x\in V$. We denote this representation by $\rho^{(r)}$. Since $G$ is reductive, it is the direct sum of irreducible representations: $\rho^{(r)}=\bigoplus_{i}\rho_{i}^{(r)}$. We denote by $W_{i}^{(r)}$ the representation space of $\rho_{i}^{(r)}: S^{r}(V)=\bigoplus_{i}W_{i}^{(r)}$. Note that a homogeneous polynomial $f(x)$ is a relative invariant of degree $r$ if and only if $f(x)\in W_{i}^{(r)}$ for some $W_{i}^{(r)}$ satisfying $\dim W_{i}^{(r)}=1$. We say that $\rho_{i}^{(r)}$ decomposes to $\rho_{j}^{(r_1)}\times\rho_{k}^{(r_2)}$ $(r_1+r_2=r)$ and denote this relation by $\rho_{i}^{(r)}\sim\rho_{j}^{(r_1)}\times\rho_{k}^{(r_2)}$ when $\rho_{i}^{(r)}$ is one of the irreducible components of the symmetric tensor of $\rho_{j}^{(r_1)}$ and $\rho_{k}^{(r_2)}$. This implies that the polynomials $\phi$ in $W_{i}^{(r)}$ can be obtained from those in $W_{i}^{(r_1)}$ and $W_{k}^{(r_2)}$, i.e., $\phi=\sum_{t}\psi_t\theta_t$ for some $\psi_t\in W_{j}^{(r_1)}$ and $\theta_t\in W_{k}^{(r_2)}$. In such a way, we can reduce the problem of

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