The maximal large sieve

Dedicated to Professor Yoshie Katsurada on the occasion of her sixtieth anniversary

By Saburô UCHIYAMA

Let M and N be integers with $N>0$ and let $a_{M+1},\cdots,a_{M+N}$ be any real or complex numbers. Define

$$S(t)=\sum_{M<n\leqq M+N} a_n e(nt)$$

with the abbreviation $e(t)=e^{2\pi it}$ and set

$$Z=\sum_{M<n\leqq M+N} |a_n|^2 .$$

Let $x_1,\cdots,x_R$ $(R\geqq 1)$ be any fixed real numbers which satisfy the condition

$$\|x_u-x_v\|\geqq\delta \qquad \text{when} \quad u\neq v,$$

where $\|x\|$ denotes the absolute distance between x and the nearest integer to it, and $0<\delta\leqq 1/2$.

In a recent paper [1] E. Bombieri and H. Davenport proved that

$$(1)\qquad \sum_{r=1}^{R}|S(x_r)|^2\leqq\begin{cases}(N^{1/2}+\delta^{-1/2})^2Z\\ 2\max(N,\delta^{-1})Z\end{cases}$$

and essentially the best possible results of the type (1) have also been obtained by them in [2]. On the other hand, P. X. Gallagher [3] has given a very simple and ingenious proof of the inequality

$$(2)\qquad \sum_{r=1}^{R}|S(x_r)|^2\leqq(\pi N+\delta^{-1})Z,$$

which is slightly weaker than, but as powerful as, (1).

Now, our principal objective in this paper is to replace in these inequalities the sum $S(t)$ by the 'maximal function' $S^*(t)$ defined by

$$S^*(t)=\sup_{1\leqq n\leqq N}\Big|\sum_{M<m<M+n} a_m e(mt)\Big|.$$

Indeed, we can show that for $N\geqq 2$

$$(3)\qquad \sum_{r=1}^{R}\big(S^*(x_r)\big)^2\leqq B(N\log N+\delta^{-1}\log^2 N)Z,$$