On a characteristic property of Sasakian manifolds with constant φ-holomorphic sectional curvarture

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction. É. Kosmanek has studied [2] a characteristic property of Kähler manifolds of constant holomorphic sectional curvature. In this paper, we shall show the following theorem making use of an analogous method of in [2].

THEOREM. *Let $M^{2n+'}$, $n \geqq 2$, be a Riemannian manifold with Sasakian structure* (ξ, φ, g). *Assume the property* (P) *to be valid in* M^{2n+1}:

(P); *For each point p of M^{2n+1} and every geodesic $\gamma(t)$ starting from p whose velocity vector at p is orthogonal to ξ_p, there exist functions $f(t)$ and $h(t)$ such that $f\varphi\gamma' + h\xi$ is a Jacobi field along γ and $f(0) \neq 0$.*

Then M is a space of constant φ-holomorphic sectional curvature.

Conversely, a Sasakian space of constant φ-holomorphic sectional curvature satisfies the property (P).

Here and throughout the paper, t means an affine parameter.

§1. Lemmas. Let (M^{2n+1}, g) be a Riemannian space. A unit Killing vector field in M is called a Sasakian structure if it satisfies

$$(\nabla_X \varphi) Y = g(\xi, Y) X - g(X, Y)\xi, \qquad \text{where} \quad \varphi X = \nabla_X \xi. \tag{1.1}$$

A Sasakian manifold is a Riemannian manifold which admits a Sasakian structure. In such a space, we know

$$R(\xi, X) Y = g(X, Y)\xi - g(\xi, Y) X. \tag{1.2}$$

We define the subspace D_p of $T_p(M)$ by $D_p = \{X \mid g(\xi, X) = 0,\ X \in T_p(M)\}$.

LEMMA 1. *Let M be a Sasakian manifold and γ be a geodesic. If the velocity vector γ' of γ at a point p is orthogonal to ξ_p, then γ' is orthogonal to ξ on γ.*

Lemma 1 follows from $\nabla_{\gamma'}(g(\xi, \gamma')) = g(\varphi\gamma', \gamma') = 0$.

LEMMA 2. *Assume that a Sasakian space M satisfies (P). Then, for vectors $X, Y \in D_p$ such that $g(\varphi X, Y) = 0$,* we have

$$g\Big(R(X, \varphi X) X, Y\Big) = 0.$$