On some 3-dimensional Riemannian manifolds

By Kouei SEKIGAWA

1. Introduction. The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

$$(*) \qquad R(X, Y)\cdot R = 0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition on the curvature tensor field R imply that $\nabla R=0$? K. Nomizu conjectured that the answer is positive in the case where (M, g) is complete irreducible and dim $M \geqq 3$. But, recently, H. Takagi [9] gave an example of 3-dimensional complete, irreducible real analytic Riemannian manifold (M, g) satisfying $(*)$ and $\nabla R \neq 0$ as a hypersurface in a 4-dimensional Euclidean space E^4. Furthermore, the present author proved that, in an $(m+1)$-dimensional Euclidean space $E^{m+1} (m \geqq 4)$, there exist some complete, irreducible real analytic hypersurfaces which satisfy $(*)$ and $\nabla R \neq 0$ ([6] in references). Let R_1 be the Ricci tensor of (M, g). Then, $(*)$ implies in particular

$$(**) \qquad R(X, Y)\cdot R_1 = 0 \qquad \text{for all tangent vectors } X \text{ and } Y.$$

In the present paper, with respect to this problem, we shall give an affirmative answer in the case where (M, g) is a certain 3-dimensional compact, irreducible real analytic Riemannian manifold, that is

THEOREM. *Let (M, g) be a 3-dimensional compact, irreducible real analytic Riemannian manifold satisfying the condition $(*)$ (or equivalently $(**)$). If the Ricci form of (M, g) is non-zero, positive semi-definite on M, then (M, g) is a space of constant curvature.*

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2. Lemmas. *Let (M, g) be a 3-dimensional real analytic Riemannian manifold. Let R^1 be a field of symmetric endomorphism satisfying $R_1(X, Y) = g(R^1X, Y)$. It is known that the curvature tensor of (M, g) is given by*

$$(2.1) \qquad R(X, Y) = R^1X \wedge Y + X \wedge R^1Y - \frac{\text{trace } R^1}{2} X \wedge Y,$$

for all tangent vectors X and Y.