On the nilpotency index of the radical of a group algebra

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Throughout the present note, K will represent *an algebraically closed field of characteristic* $p>0$. In case G is a p-solvable group of order $p^a m$ $(a\geqq 1, p\nmid m)$, concerning the nilpotency index $t(G)$ of the radical $J(KG)$ of the group algebra KG, D. S. Passman [4; Th. 1. 6], Y. Tsushima [5; Th .2] and D. A. R. Wallace [7; Th. 3. 3] have obtained the following:

$$p^a \geqq t(G) \geqq a(p-1)+1 .$$

In §§1 and 2 of the present note, we shall investigate when $t(G)=p^a$ or $t(G)=a(p-1)+1$, where G is a p-solvable group of order $p^a m (a\geqq 1, p\nmid m)$. Furthermore, as an application of Th. 1, we shall present a characterization of a finite group G with $t(G)=[J(KG): K]+1$ (Th. 2).

1. We shall begin our study with the following:

THEOREM 1. *If G is a p-group of order p^a, then there holds the following*:

(1) $t(G)=a(p-1)+1$ *if and only if G is elementary abelian.*

(2) $t(G)=p^a$ *if and only if G is cyclic.*

PROOF. (1) Following [3], we consider the $\mathfrak{K}$-series of G:

$$G=\mathfrak{K}_1 \supseteq \mathfrak{K}_2 \supseteq \cdots \supseteq \mathfrak{K}_{t(G)}=1 ,$$

where $\mathfrak{K}_\lambda=\{x\in G|1-x\in J(KG)^\lambda\}$. Then, every $\mathfrak{K}_\lambda$ is a characteristic subgroup of G and $\mathfrak{K}_\lambda/\mathfrak{K}_{\lambda+1}$ is an elementary abelian group of order p^{d_λ}. By [3; Th. 3. 7], we have $t(G)=\sum_\lambda \lambda d_\lambda(p-1)+1$. If $t(G)=a(p-1)+1$ then $\sum_\lambda \lambda d_\lambda=a$. Combining this with $\sum_\lambda d_\lambda=a$, we readily obtain $d_1=a$ and $d_\lambda=0 (\lambda\neq 1)$, namely, G is elementary abelian. The converse is obvious by [3; Th. 6. 2].

(2) Suppose $t(G)=p^a$. If $\Phi(G)$ is the Frattini subgroup of G, then [7; Th. 2. 4] yields $|G|=t(G)\leqq t(\Phi(G))\cdot t(G/\Phi(G))\leqq |\Phi(G)|\cdot|G/\Phi(G)|=|G|$, whence it follows $t(G/\Phi(G))=|G/\Phi(G)|=p^b (b\leqq a)$. Since $G/\Phi(G)$ is elementary abelian, $t(G/\Phi(G))=b(p-1)+1$ by (1). Hence, $p^b=|G/\Phi(G)|=t(G/\Phi(G))=b(p-1)+1$, which means $b=1$ and $G/\Phi(G)$ is cyclic. Now, as is well-known, G is cyclic. Concerning the converse, there is nothing to prove.

In what follows, G_p will represent a Sylow p-subgroup of G.