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Cauchy problems for the operator of Tricomi's type

By Masato IMAI

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§ 1. Introduction.

In this paper we shall consider the well-posedness of the Cauchy problem for the weakly hyperbolic partial differential operator of the second order such that:

$$(1.1)\qquad P(x, D)=D_{x_0}^2-x_0 A(x, x_{x'})+P_1(x, D_x),$$

where $D_x=\left(\frac{1}{i}\frac{\partial}{\partial x_0}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$, $A(x, D_{x'})$ is an elliptic operator whose principal symbol is positive and P_1 is an arbitrary first order term. We treat $P(x, D)$ in the closed half space $\overline{R_+^{n+1}}=\{x;\ x=(x_0, x'),\ x'=(x_1, \cdots, x_n),\ x_0\geqq 0\}$ and assume that the coefficients of $P(x, D)$ are constant for large $|x'|$. Note that the principal symbol $P_2(x, \xi)$ of P has no critical point with respect to (x, ξ) i. e., $grad_{(x,\xi)} P_2(x^0, \xi^0)\neq 0$ for any $(x^0, \xi^0), \xi^0\neq 0$.

Oleinik proved the well-posedness of (1.1) by the energy estimate ([11]). More generally, Ivrii proved that if the principal symbol of the weakly hyperbolic operator has no critical point, then the Cauchy problem is well-posed for an aribitrary lower order term and called these operators completely regularly hyperbolic ([5], [6]). He also proved the above fact by the energy estimate. However, in this paper, using Airy function we shall construct the fundamental solution of (1.1) and give somewhat sharp results.

Airy function was used in the construction of the parameterix for the exterior mixed problem for hyperbolic equations at a diffractive point ([2], [4], [8], [13]). The situation is similar if the boundary surface is replaced by the initial one, because the bicharacteristic strip $(x(t), \xi(t))$ of P_2 is tangent to the hyper-surface $x_0=0$ in $\overline{R_{+,x}^{n+1}}\times(R_\xi^{n+1}\backslash 0)$ of order one.

Let us formulate our problem more precisely as follows: for given data $f(x)$, $v_0(x')$ and $v_1(x')$, we shall consider the Cauchy problem with initial surface $x_0=0$ such that;

$$(1.2)\qquad \begin{cases} P(x, D)\, u=f(x) & \text{in } [0, T]\times R^n, \\ D_{x_0}^j u|_{x_0=0}=v_j(x') & \text{in } R^n (j=0, 1), \end{cases}$$

where T is some positive number.