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On the infinitesimal Blaschke conjecture

By Kazuyoshi KIYOHARA

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Introduction

Let M be a riemannian manifold and g its riemannian metric. Then M is called a C_l-manifold and g a C_l-metric if all of its geodesics are periodic and have the same length l. So far very few C_l-manifolds are known except for the following famous examples: The spheres S^n $(n\geqq 1)$ and the various projective spaces, *i.e.*, the real projective spaces $\boldsymbol{RP}^n$ $(n\geqq 2)$, the complex projective spaces $\boldsymbol{CP}^n$ $(n\geqq 2)$, the quaternion projective spaces $\boldsymbol{HP}^n$ $(n\geqq 2)$, and the Cayley projective plane $\boldsymbol{CaP}^2$, all of these being equipped with the standard metrics. In the case of S^n we know that there are non-standard C_l-metrics, which are given by Zoll and Weinstein (cf. [1]). On the other hand, for $\boldsymbol{RP}^n$, the non-existence of such metrics was proved by Berger (cf. [1] Appendix D). But it is not known whether there exist non-standard C_l-metrics on any other projective space. For a historical reason, the conjecture of non-existence of such metrics on the projective spaces is called the Blaschke conjecture.

The main purpose of the present paper is to study an infinitesimal version of the Blaschke conjecture, the infinitesimal Blaschke conjecture, and to give a partial affirmative answer to this conjecture.

Let M be one of the spaces $\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, and $\boldsymbol{CaP}^2$, and g_0 its standard C_π-metric. Let us consider a deformation g_t of the riemannian metric g_0 which satisfies the following conditions:

1) Each g_t is a C_π-metric;

2) Each g_t is semi-conformal to g_0, *i.e.*, for any projective line $N\subset M$ there is a function h_t on N such that $\iota^* g_t = h_t \iota^* g_0$, where ι denotes the inclusion $N\to M$.

Then we know that the linearization $f=\left.\dfrac{\partial g_t}{\partial t}\right|_{t=0}$ of g_t at $t=0$, being a symmetric 2-form on M, satisfies the following conditions:

a) $\displaystyle\int_0^\pi f\big(\dot\gamma(t), \dot\gamma(t)\big)\, dt=0$ for any geodesic $\gamma(t)$ with $\|\dot\gamma(t)\|=1$;

b) f is semi-conformal to g_0, *i.e.*, for any projective line $N\subset M$ there is a function h on N such that $\iota^* f = h\iota^* g_0$, where ι denotes the inclusion