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# On pseudo-product graded Lie algebras

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## Introduction.

For several years, N. Tanaka has worked on the geometry of pseudo-product manifolds in connection with the geometric study of systems of $k$-th order ordinary differential equations, where $k\geqq 2$. A study in this line can be found in his recent paper [6]. His theory shows that the geometry is closely related to the study of pseudo-product graded Lie algebras, which we will explain later on.

The main purpose of this paper is to prove structure theorems on some restricted types of pseudo-product graded Lie algebras.

Let $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{g}_p$ be a graded Lie algebra with $0<\dim\mathfrak{m}<\infty$. Then $\mathfrak{m}$ is called a fundamental graded Lie algebra or simply an FGLA, if $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the triplet $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ is called a pseudo-product FGLA if the following conditions are satisfied:

(1) $\mathfrak{m}$ is an FGLA.

(2) $\mathfrak{g}_{-1}=\mathfrak{e}\oplus\mathfrak{f}$ and $[\mathfrak{e},\mathfrak{e}]=[\mathfrak{f},\mathfrak{f}]=\{0\}$

A pseudo-product FGLA $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ is called non-degenerate, if the condition "$x\in\mathfrak{g}_{-1}$ and $[x,\mathfrak{g}_{-1}]=\{0\}$" implies $x=0$.

Now let $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ be a graded Lie algebra and let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Set $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{g}_p$. Then $\mathfrak{g}$ (together with $\mathfrak{e}$ and $\mathfrak{f}$) is called a pseudo-product graded Lie algebra if the following conditins are satisfied:

(1) $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ is a pseudo-product FGLA.

(2) $\mathfrak{g}$ is transitive, i.e. the condition "$p\geqq 0$, $x\in\mathfrak{g}_p$ and $[x,\mathfrak{g}_{-1}]=\{0\}$" implies $x=0$.

(3) $[\mathfrak{g}_0,\mathfrak{e}]\subset\mathfrak{e}$ and $[\mathfrak{g}_0,\mathfrak{f}]\subset\mathfrak{f}$

Let $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ be an FGLA and $\mathfrak{g}_0$ be its derivations of the graded Lie algebra $\mathfrak{m}$ leaving both $\mathfrak{e}$ and $\mathfrak{f}$ invariant. Then the prolongation $\check{\mathfrak{g}}=\bigoplus_{p\in Z}\check{\mathfrak{g}}_p$ of the pair $(\mathfrak{m}\,;\mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ (see [4] and [6]), which may be characterized as the maximum pseudo-product graded Lie algebra $\mathfrak{g}=\bigoplus_{p\in Z}\mathfrak{g}_p$ such that $\bigoplus_{p\leq 0}\mathfrak{g}_p=\mathfrak{m}\oplus\check{\mathfrak{g}}_0$ (as graded Lie algebras). It is known that if $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ is non-degenerate, then $\check{\mathfrak{g}}$ is of finite dimension (see