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Theorem of Busemann-Mayer on Finsler metrics

To Professor Noboru Tanaka on his sixtieth birthday

Shoshichi KOBAYASHI*
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1. Introduction

Let X be a manifold and TX its tangent bundle. A *pseudo-length function* on X is a real valued nonnegative function F on TX satisfying the condition

(1) $$F(c\xi)=|c|F(\xi) \qquad \text{for } \xi\in TX,\ c\in\mathbf{R}.$$

If $F(\xi)>0$ for every nonzero ξ, then F is called a *length function.*

If $\xi\in T_xX$, we write sometimes (x, ξ) for ξ although x is redundant. Similarly, we write occasionally $F(x, \xi)$ for $F(\xi)$. When we are working in a coordinate neighborhood U with a natural identification $TU\simeq U\times\mathbf{R}^n$, the notation $F(x, \xi)$ is more convenient as well as traditional since ξ may be used to denote an element of $\mathbf{R}^n$ as well as an element of TU.

We say that F is *convex* if it defines a pseudo-norm on each tangent space $T_xX, x\in X$, i e., if

(2) $$F(\xi+\xi')\leq F(\xi)+F(\xi') \qquad \text{for } \xi, \xi'\in T_xX.$$

A convex length function is usually called a *Finsler metric.*

Given a pseudo-length function F, its *indicatrix* Γ_x at $x\in X$ is defined to be

(3) $$\Gamma_x=\{\xi\in T_xX\,;\, F(\xi)\leq 1\}.$$

Then Γ_x is (1) star shaped in the sense that if $\xi\in\Gamma_x$ then $c\xi\in\Gamma_x$ for $|c|\leq 1$ and is (2) nontrivial in every direction in the sense that for every $\xi\in T_xX$ there is a nonzero c such that $c\xi\in\Gamma_x$.

Conversely, given a subset Γ_x in each tangent space T_xX satisfying the two conditions above, we can construct a pseudo-length function F by

(4) $$F(\xi)=\inf\{c>0\,;\,\frac{\xi}{c}\in\Gamma_x\} \qquad \text{for } \xi\in T_xX.$$

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