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Matrix invariants of binary forms

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Abstract. Let S_n be the vector space of homogeneous polynomials of degree n in two variables. Let $A_d(n)$ be the noncommutative algebra consisting of SL_2-equivariant polynomial maps from S_d to $\mathrm{End}S_n$. We show that generators for $A_d(n)$ are derived from generators for the algebra of covariants of the d-ic forms.

Key words: binary forms, covariants, transvectants, Clebsch-Gordan rule.

Introduction

Let k be a field of characteristic 0. Put $S = k[x_1, x_2]$, the polynomial ring, and let S_n be its homogenous part of degree n. The group SL_2 acts on S canonically. We are concerned about SL_2-invariant polynomial maps from the space S_d to the matrix algebra $\mathrm{End}S_n$. Those maps form an algebra $A_d(n)$ by matrix multiplication. $A_d(0)$ is the algebra of invariants of the d-ic form, and was studied in classical invariant theory. We show that $A_d(n)$ is a deformation of a factor of the algebra of covariants of the d-ic form. In particular, the knowledge of the generators for the algebra of covariants gives that for the algebra $A_d(n)$.

More generally, let R be a commutative algebra with SL_2-action. Then SL_2 acts on the algebra $R \otimes \mathrm{End}S_n$ and let $A(n) = (R \otimes \mathrm{End}S_n)^{SL_2}$ be the invariant algebra. On the other hand, we have the commutative algebra $C = (R \otimes S)^{SL_2}$ with grading given by $C_n = (R \otimes S_n)^{SL_2}$. For $\alpha \in R \otimes S_n$, $\beta \in R \otimes S_m$ and $p \geq 0$, we have the transvectant $(\alpha, \beta)_p \in R \otimes S_{n+m-2p}$, where α, β are regarded as forms with coefficients in R ([1]). Define the map $\phi : \bigoplus_{p=0}^{n} C_{2p} \to A(n)$ by

$$\phi(\alpha)(\gamma) = \frac{n!}{(n-p)!}(\alpha, \gamma)_p$$

for $\alpha \in C_{2p}$, $\gamma \in R \otimes S_n$. Then it is shown that ϕ is an isomorphism and

$$\phi(\alpha)\phi(\beta) - \phi(\alpha\beta) \in \phi(\bigoplus_{r<p+q} C_{2r})$$

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