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Equilibrium vector potentials in $\mathbb{R}^3$

(Dedicated to Professor Makoto Ohtsuka on his 70th birthday)

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Abstract. In the potential theory it is well known that the notion of equilibrium potentials for a bounded domain D with smooth boundary surfaces Σ in $\mathbb{R}^3$ is rested on the basis of the electric condenser. In this paper we introduce the notion of equilibrium vector potentials for D based on the electric solenoid. We then find that such vector potentials are related to harmonic 2-forms on $\overline{D}$ whose normal component with respect to Σ vanishes at any point of Σ.

Key words: vector potential, solenoid, harmonic forms, newton kernel, weyls' orthogonal decomposition.

Introduction

Let D be an electric condenser with smooth boundary surfaces Σ in $\mathbb{R}^3$. Then D carries the equilibrium charge distribution $\rho\, dS_x$ on Σ, where dS_x is the surface area element of Σ, which induces the electric field $E(x)$ in $\mathbb{R}^3 \setminus \Sigma$ being identically 0 in D:

$$u(x) = \frac{1}{4\pi}\int_\Sigma \frac{\rho(y)}{\|x-y\|} dS_y \qquad \text{for } x \in \mathbb{R}^3,$$

$$E(x) = \text{grad } u(x) = \frac{-1}{4\pi}\int_\Sigma \rho(y)\frac{x-y}{\|x-y\|^3} dS_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma.$$

The function $u(x)$ is called the equilibrium potential for D. We consider the total energy $\mu = \int_{\mathbb{R}^3} \|E(x)\|^2\, dv_x$ of the electric field $E(x)$. Now assume that the condenser D_t varies smoothly with real parameter t. Then the total energy $\mu(t)$ varies with parameter t. In [Y1, §2] and [LY, §9], we formed the variation formula of second order $\mu''(t)$ for $\mu(t)$ with respect to t. We intend to make the corresponding studies in the magnetic fields' version. In this paper, motivated by the electric solenoid (see the beginning of §8) we introduce the notion of equilibrium current density JdS_x on Σ, the magnetic field $B(x)$ induced by JdS_x and the equilibrium vector potential $A(x)$, and study their properties. We consider the total energy $\nu = \int_{\mathbb{R}^3} \|B(x)\|^2 dv_x$ of

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