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On the scattering theory for the cubic nonlinear Schrödinger and Hartree type equations in one space dimension

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Abstract. We study the scattering problem and asymptotics for large time of solutions to the Cauchy problem for the nonlinear Schrödinger and Hartree type equations with subcritical nonlinearities

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = f(|u|^2)u, & (t,x) \in \mathbf{R}^2 \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where the nonlinear interaction term is $f(|u|^2) = V * |u|^2$, $V(x) = \lambda|x|^{-\delta}$, $\lambda \in \mathbf{R}$, $0 < \delta < 1$ in the Hartree type case, or $f(|u|^2) = \lambda|t|^{1-\delta}|u|^2$ in the case of the cubic nonlinear Schrödinger equation. We suppose that the initial data $e^{\beta|x|}u_0 \in L^2$, $\beta > 0$ with sufficiently small norm $\epsilon = \|e^{\beta|x|}u_0\|_{L^2}$. Then we prove the sharp decay estimate $\|u(t)\|_{L^p} \leq C\epsilon t^{\frac{1}{p}-\frac{1}{2}}$, for all $t \geq 1$ and for every $2 \leq p \leq \infty$. Furthermore we show that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $\hat{u}_+ \in L^2$ such that for all $t \geq 1$

$$\|u(t) - \exp\Big(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big)U(t)u_+\|_{L^2} = O(t^{1-2\delta})$$

and uniformly with respect to x

$$u(t,x) = \frac{1}{(it)^{\frac{1}{2}}}\hat{u}_+\Big(\frac{x}{t}\Big)\exp\Big(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\Big(\frac{x}{t}\Big)\Big) + O(t^{1/2-2\delta}),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ. Our results show that the regularity condition on the initial data which was assumed in the previous paper [9] is not needed. Also a smoothing effect for the solutions in an analytic function space is discussed.

Key words: nonlinear Schrödinger, scattering, subcritical case.

1. Introduction

We study the asymptotic behavior for large time of solutions to the Cauchy problem

$$\begin{cases} i\partial_t u = -\frac{1}{2}\partial_x^2 u + f(|u|^2)u, & (t,x) \in \mathbf{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

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