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CL-shellability of ordered structure of reflection systems

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Abstract. We define a "Bruhat" order on a reflection system, and show that each closed interval is CL-shellable. The reflection systems, introduced by M. Dyer, generalize the Coxeter systems. Therefore our result is a generalization of a result of A. Björner and M. Wachs for Coxeter groups.

Key words: Coxeter groups, Bruhat orders, CL-shellability, Cohen-Macaulay posets.

1. Introduction

The lexicographic shellability of posets was first introduced by Björner [Bj]. If a poset P is CL-shellable, then P is Cohen-Macaulay over an arbitrary field [BGS]. Cohen-Macaulay posets have been one of the main interests in combinatorics and have been studied deeply from combinatorial, algebraic, and topological points of view [S2] [Bac] [H] [M1] [M2]. However, directly from its algebraic or geometric definition, it is sometimes difficult to see whether a given poset is Cohen-Macaulay or not. On the other hand, the definition of CL-shellability is completely combinatorial. Consequently the CL-shellabitity makes it easier to see whether or not a given poset is Cohen-Macaulay than the original definitions. One of the most remarkable classes of CL-shellable ordered structures is the Bruhat order of Coxeter systems. CL-shellability of the Bruhat order was first proved for the symmetric groups by Edelman [Ed], and for the classical Weyl groups by Proctor [Pr]. Finally Björner and Wachs showed it for any Coxeter groups [BW1].

The purpose of the present paper is to extend the result of Björner-Wachs for Coxeter systems to reflection systems. The reflection systems were first introduced by Dyer [D] in his study of reflection subgroups of Coxeter groups. Reflection systems are a generalized notion of Coxeter systems. The most noticeable difference lies in the orders of generators: In the case of a Coxeter group, we can choose a system of generators consisting of involutions of order two. On the other hand orders of generators of a reflection system can be an arbitrary nonnegative integer (or even the

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