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# Weierstrass's function and chaos

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In this paper, we discuss the relationship between the famous Weierstrass's everywhere non-differentiable continuous function and the chaotic dynamical system. With this formulation, we can find an equation for which Weierstrass's function is a solution. This is a trial to combine two notions: Fractals and Chaos. B. B. Mandelbrot introduced the first object in his book [1] but he mentioned that he do not know how relate these two notions.* Of course, we owe many nice ideas to this Mandelbrot's book.

## 1. Introduction

It was proved by Weierstrass that the function:

$$W_{a,b}(x)=\sum_{n=0}^{\infty} a^{n}\cos(b^{n}\pi x) \tag{1.1}$$

where $0<a<1$, $b$ is an odd integer and $ab>1+\frac{3}{2}\pi$, has no differential coefficient for any value of $x$. Weierstrass's result was generalized by G. H. Hardy [2], who has proved that $W_{a,b}(x)$ does not possess a finite differential coefficient at any point $x$ in any case in which $0<a<1$, $b>1$ and $ab\geq 1$.

Our starting point is just in the representation of $\Psi^{n}(x)$, which is an $n$-fold iteration by $\Psi(x)=4x(1-x)$, that is,

$$\Psi^{n}(x)=\sin^{2}(2^{n}\,\mathrm{Arcsin}\sqrt{x})\,. \tag{1.2}$$

This function $\Psi(x)$ is called chaotic in the sense of Li-Yorke [10].
Combining (1.1) and (1.2), we have

$$\sum_{n=0}^{\infty} t^{n}\Psi^{n}(x)=\frac{1}{2(1-t)}-\frac{1}{2}W_{t,2}\left(\frac{2}{\pi}\,\mathrm{Arcsin}\sqrt{x}\right). \tag{1.3}$$

Perhaps one should think the left hand side of (1.3) as a generating function which generates the iterations of $\Psi$. More generally, it will be very interesting and important to investigate the function:

$$F(t,x)=\sum_{n=0}^{\infty} t^{n}g\big((\varphi^{n}(x)\big) \tag{1.4}$$