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How to Determine the *Sign* of a Valuation on $\mathbb{C}[x, y]$

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Abstract. Given a divisorial discrete valuation *centered at infinity* on $\mathbb{C}[x, y]$, we show that its sign on $\mathbb{C}[x, y]$ (i.e. whether it is negative or nonpositive on $\mathbb{C}[x, y] \setminus \mathbb{C}$) is completely determined by the sign of its value on the *last key form* (key forms being the avatar of *key polynomials* of valuations [Mac36] in "global coordinates"). We also describe the cone of curves and the nef cone of certain compactifications of $\mathbb{C}^2$ associated with a given valuation centered at infinity and give a characterization of the divisorial valuations centered at infinity whose *skewness* can be interpreted in terms of the *slope* of an extremal ray of these cones, yielding a generalization of a result of [FJ07]. A byproduct of these arguments is a characterization of valuations that "determine" normal compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity in terms of an associated "semigroup of values".

1. Introduction

Notation 1.1. Throughout this section, k is a field, and R is a finitely generated k-algebra.

In algebraic (or analytic) geometry and commutative algebra, valuations are usually treated in the *local setting,* and the values are always positive or nonnegative. Even if it is a priori not known if a given discrete valuation ν is positive or nonnegative on $R \setminus k$, it is evident how to verify this, at least if $\nu(k \setminus \{0\}) = 0$: we have only to check the values of ν on the k-algebra generators of R. For valuations *centered at infinity* however, in general, it is nontrivial to determine if it is negative or nonpositive on $R \setminus k$:

Example 1.2. Let $R := \mathbb{C}[x, y]$, and for every $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$, let ν_ε be the valuation (with values in $\mathbb{R}$) on $\mathbb{C}(x, y)$ defined as follows:

$$\begin{aligned} &\nu_\varepsilon(f(x, y)) := -\deg_x(f(x, y)|_{y=x^{5/2}+x^{-1}+\xi x^{-5/2-\varepsilon}}) \\ &\qquad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\}, \end{aligned} \tag{1}$$

where ξ is a new indeterminate, and $\deg_x$ is the degree in x. A direct computation shows that

$$\nu_\varepsilon(x) = -1, \qquad \nu(y) = -5/2,$$
$$\nu_\varepsilon(y^2 - x^5) = -3/2, \qquad \nu_\varepsilon(y^2 - x^5 - 2x^{-1}y) = \varepsilon.$$

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