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# Universality in Measure in the Bulk for Varying Weights

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Abstract. We prove that universality holds in measure for sequences $\{g_n e^{-2nQ}\}$ of varying weights, where $e^{-Q}$ is an exponential weight, and the functions $\{g_n\}$ admit suitable polynomial approximations.

## 1. Introduction

In the theory of random Hermitian matrices, one considers a probability distribution $\mathcal{P}^{(n)}$ on the eigenvalues $x_1 \le x_2 \le \cdots \le x_n$ of $n \times n$ Hermitian matrices,

$$\mathcal{P}^{(n)}(x_1, x_2, \dots, x_n) = ce^{-\sum_{j=1}^{n} 2nQ_n(x_j)} \prod_{i<j} (x_i - x_j)^2.$$

See [5, p. 106 ff.]. Here, $c$ is a normalizing constant, often called the partition function, and $Q_n$ is a given function. In the Gaussian unitary ensemble, $Q_n(x) = \frac{1}{2}x^2$.

Orthogonal polynomials play a crucial role in analyzing these. For $n \ge 1$, let $\mu_n$ be a finite positive Borel measure with support supp$[\mu_n]$ containing infinitely many points. We may define orthonormal polynomials

$$p_m(\mu_n, x) = \gamma_m(\mu_n)x^m + \cdots, \quad \gamma_m(\mu_n) > 0,$$

$m = 0, 1, 2, \dots$, satisfying the orthonormality conditions

$$\int p_k(\mu_n, x) p_\ell(\mu_n, x)\, d\mu_n(x) = \delta_{k\ell}.$$

Throughout we use $\mu_n'$ to denote the Radon–Nikodym derivative of $\mu_n$. The $n$th reproducing kernel for $\mu_n$ is [10; 23]

$$K_n(\mu_n, x, y) = \sum_{k=0}^{n-1} p_k(\mu_n, x) p_k(\mu_n, y), \tag{1.1}$$

and the normalized kernel is

$$\tilde{K}_n(\mu_n, x, y) = \mu_n'(x)^{1/2} \mu_n'(y)^{1/2} K_n(\mu_n, x, y). \tag{1.2}$$

The $n$th Christoffel function is

$$\lambda_n(\mu_n, x) = K_n(\mu_n, x, x)^{-1}.$$

When

$$d\mu_n(x) = e^{-2nQ_n(x)}\, dx,$$

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