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Pretzel Knots with L-Space Surgeries

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Abstract. A rational homology sphere whose Heegaard Floer homology is the same in rank as that of a lens space is called an L-space. We classify pretzel knots with any number of tangles that admit L-space surgeries. This rests on Gabai's classification of fibered pretzel links.

1. Introduction

The Heegaard Floer homology of three-manifolds and its refinement for knots, knot Floer homology, have proved to be particularly useful for studying Dehn surgery questions in three-manifold topology. Recall that the knot Floer homology of a knot K in the three-sphere is a bigraded Abelian group,

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(K, s),$$

introduced by Ozsváth and Szabó [OS04b] and independently by Rasmussen [Ra03]. The graded Euler characteristic is the symmetrized Alexander polynomial of K [OS04b],

$$\Delta_K(t) = \sum_s \chi(\widehat{\mathrm{HFK}}(K, s)) \cdot t^s.$$

These theories have been especially useful for studying knots that admit lens space surgeries, the classification of which has been an outstanding problem in low-dimensional topology for decades. For example, if $K \subset S^3$ admits a lens space surgery, then for all $s \in \mathbb{Z}$, we have $\widehat{\mathrm{HFK}}(K, s) \cong 0$ or $\mathbb{Z}$ [OS05, Thm. 1.2]. Knot Floer homology detects both the genus of K by

$$g(K) = \max\{s \mid \widehat{\mathrm{HFK}}(K, s) \neq 0\}$$

[OS04a] and the fiberedness of K by whether $\widehat{\mathrm{HFK}}(K, g(K))$ is isomorphic to $\mathbb{Z}$ [Ghi08; Ni07]. Together, these facts imply that a knot in S^3 with a lens space surgery is fibered. Indeed, this result applies more generally to knots in S^3 admitting L-space surgeries. Recall that a rational homology sphere Y is an *L-space* if $|H_1(Y; \mathbb{Z})| = \operatorname{rank} \widehat{\mathrm{HF}}(Y)$, where $\widehat{\mathrm{HF}}$ is the "hat" flavor of Heegaard Floer homology. The class of L-spaces includes all lens spaces, and more generally, three-manifolds with elliptic geometry [OS05, Prop. 2.3] (or equivalently, with finite fundamental group by the Geometrization theorem; see [KL08]). A knot admitting an L-space surgery is called an *L-space knot*.

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