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Hilbert Transform Characterization and Fefferman–Stein Decomposition of Triebel–Lizorkin Spaces

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1. Introduction

There are several equivalent definitions for $H^1(\mathbb{R}^n)$. One of these is the Riesz transforms characterization (cf. [10, p. 221]); that is, $H^1(\mathbb{R}^n)$ consists of the class of $L^1(\mathbb{R}^n)$ functions such that their Riesz transforms belong to $L^1(\mathbb{R}^n)$ as well. Furthermore,

$$\|f\|_{H^1} \approx \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j(f)\|_{L^1},$$

where the R_j are the Riesz transforms. By the duality between H^1 and BMO, every $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$ can be represented as

$$\varphi = \varphi_0 + \sum_{j=1}^{n} R_j(\varphi_j) \quad \text{(modulo constants)},$$

where $\varphi_0, \varphi_j \in L^\infty(\mathbb{R}^n)$ (see [4, Thm. 3]). This decomposition is widely known as the Fefferman–Stein decomposition.

Many authors (see e.g. [1; 2; 3; 7; 8; 11]) have generalized the Riesz transforms characterization and Fefferman–Stein decomposition to different variants of Hardy spaces and BMO spaces. Since both H^1 and BMO are special cases of Triebel–Lizorkin spaces, we seek to extend these two properties to (respectively) the Triebel–Lizorkin spaces $\dot{F}_1^{0,q}(\mathbb{R})$ $(2 \le q < \infty)$ and their duals $\dot{F}_\infty^{0,q'}(\mathbb{R})$.

Let $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ denote the Schwartz space and its dual, respectively. Choose a fixed function $\varphi \in \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\varphi) \subset \{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}$, $|\hat{\varphi}(\xi)| \ge C > 0$ for $3/5 \le |\xi| \le 5/3$, and $\sum_{j\in\mathbb{Z}} |\hat{\varphi}(2^j\xi)|^2 = 1$ if $\xi \ne 0$. Write $\varphi_j(x) = 2^j\varphi(2^jx)$, $j \in \mathbb{Z}$. For $1 < q < \infty$, the homogeneous Triebel–Lizorkin space $\dot{F}_1^{0,q}(\mathbb{R})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$, the tempered distributions modulo polynomials, satisfying

$$\|f\|_{\dot{F}_1^{0,q}} := \left\| \left\{ \sum_{j\in\mathbb{Z}} (|\varphi_j * f|)^q \right\}^{1/q} \right\|_{L^1} < \infty.$$

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