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On Degree Growth and Stabilization of Three-Dimensional Monomial Maps

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1. Introduction

Given a rational self-map $f: X \dashrightarrow X$ on an n-dimensional Kähler manifold X, one can define a pullback map $f^*: H^{p,p}(X) \to H^{p,p}(X)$ for $0 \le p \le n$. In general, the pullback does not commute with iteration; that is, $(f^*)^k \ne (f^k)^*$. Following Sibony ([Si]; see also [FoSi]), we call the map f (algebraically) *stable* if the action on the cohomology of X is compatible with iterations. More precisely, f is called *p-stable* if the pullback on $H^{p,p}(X)$ satisfies $(f^*)^k = (f^k)^*$ for all $k \in \mathbb{N}$.

If f is not p-stable on X, one might try to find a birational change of coordinate $h: X' \dashrightarrow X$ such that $\tilde{f} = h^{-1} \circ f \circ h$ is p-stable. This is not always possible even for $p = 1$, as shown by Favre [Fa]. However, for $p = 1$ and $n = 2$, one can find such a stable model (with at worst quotient singularities) for quite a few classes of surface maps [DFa; FaJ]. Also for $p = 1$, such a model can be obtained for certain monomial maps [Fa; JW; L].

For an $n \times n$ integer matrix $A = (a_{i,j})$, the associated monomial map $f_A: (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is defined by

$$f_A(x_1, \dots, x_n) = \left(\prod_j x_j^{a_{1,j}}, \dots, \prod_j x_j^{a_{n,j}} \right).$$

The morphism f_A extends to a rational map, which is also denoted f_A, on any n-dimensional toric variety. The question of finding a stable model for f_A (or showing that there is no stable model for certain f_A) has been studied in [Fa; JW; L]. In particular, the stabilization problem is fully classified for dimension two in [Fa] and [JW].

In this paper, we focus on the case when $n = 3$ and A is diagonalizable. We deal with both the 1-stable and the 2-stable problems. There are more cases than dimension two. A main case where a model that is both 1-stable and 2-stable can be obtained by performing proper modification is summarized in the following theorem.

Theorem 1.1. *Let Δ be a fan in $N \cong \mathbb{Z}^3$, and let $f_A: X(\Delta) \dashrightarrow X(\Delta)$ be the monomial map associated to A. Suppose that A is diagonalizable and that, for each eigenvalue μ of A, $\mu/\bar{\mu}$ is a root of unity. Then there exists a complete*

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