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Restricted Carathéodory Measure and Restricted Volume of the Canonical Bundle

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1. Introduction

This paper is concerned with relations between Carathéodory measure hyperbolicity and algebro-geometric positivity of the canonical bundle or the cotangent bundle over a compact complex manifold. On the one hand, positivity of vector bundles over a compact complex manifold is an important notion in algebraic geometry. On the other hand, Carathéodory measure hyperbolicity for a complex manifold is one of the principal properties in geometric function theory or the theory of hyperbolic complex spaces. It is therefore of fundamental interest to investigate how these notions are related.

In the author's previous paper [20] it is proved that the curvature function of the Carathéodory pseudo-volume form over a complex manifold is not larger than -1. As an easy application of the curvature property, we obtain the following explicit comparison formula between the volume $\mathrm{vol}_X(K_X)$ of the canonical bundle K_X over a compact complex manifold X and the total volume of X with respect to the Carathéodory measure $\mu^C_{\tilde{X}}$ (see Definition 2.1) of its universal cover $\tilde{X}$.

Theorem 1.1 [20, Cor. 1.2]. *Let X be an n-dimensional compact complex space with at most normal singularities, and let $\tilde{X}$ be its universal covering space. Then*

$$\mathrm{vol}_X(K_X) := \limsup_{m\to\infty} \frac{\dim \mathrm{H}^0(X, \mathcal{O}(mK_X))}{m^n/n!} \geq \frac{n!\,(n+1)^n}{(4\pi)^n}\mu^C_{\tilde{X}}(X),$$

where the Carathéodory measure $\mu^C_{\tilde{X}}$ of $\tilde{X}$ is considered as a measure on X.

Note that differently from the original theorem in [20], the complex space in this theorem is allowed to have at most normal singularities, but the point can be solved easily by taking a resolution of the singularities of X.

This result tells us not only that the Carathéodory measure hyperbolicity of the universal cover (i.e., $\mu^C_{\tilde{X}}(X) > 0$) implies the bigness of the canonical bundle (i.e., $\mathrm{vol}_X(K_X) > 0$) but also how the bigness increases as the Carathéodory measure hyperbolicity becomes stronger.

On the other hand, for a line bundle L over a compact complex manifold X, a restricted version of the volume of L recently appears as an algebro-geometric

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