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Topological Recursion for Symplectic Volumes of Moduli Spaces of Curves

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1. Introduction

Since Kontsevich's proof [21] of the Witten conjecture [39], there has been a flurry of activity centered around the tautological ring of the moduli space of curves and expanded more generally to Gromov–Witten invariants. However, many of the fundamental tools developed by Kontsevich have remained comparatively ignored.

In this paper we focus on the combinatorially defined 2-form $\Omega_{\mathbf{L}}$ used by Kontsevich to represent the scaled sum of ψ-classes

$$[\Omega_{\mathbf{L}}] = \frac{1}{2}(L_1^2\psi_1 + \cdots + L_n^2\psi_n).$$

In particular, this form leads to a family of symplectic structures on the moduli space of curves, with the associated volumes encoding all possible ψ-class intersection numbers. Although the nondegeneracy of Ω appeared in Kontsevich's original work, the symplectic nature of Ω was not exploited in any particular way.

We develop a recursive formula (an example of *topological recursion,* as explained hereafter) for calculating the symplectic volume of the moduli space of curves. In particular, if $\mathrm{Vol}_{g,n}(L_1,\dots,L_n)$ represents the symplectic volume of $\overline{\mathcal{M}}_{g,n}$ calculated with respect to the symplectic form $\Omega_{\vec{L}}$, then we have the following statement.

Theorem 1.1. *The symplectic volumes of moduli spaces of curves obey the recursion relation*

$$\begin{aligned}
&L_1\,\mathrm{Vol}_{g,n}(L_1,\dots,L_n)\\
&\quad= \sum_{j=2}^{n}\int_{|L_1-L_j|}^{L_1+L_j} dx\,\frac{x}{2}(L_1+L_j-x)\,\mathrm{Vol}_{g,n-1}(x,L_2,\dots,\hat{L}_j,\dots,L_n)\\
&\qquad+\sum_{j=2}^{n}\int_{0}^{|L_1-L_j|} dx\,xf(x,L_1,L_j)\,\mathrm{Vol}_{g,n-1}(x,L_2,\dots,\hat{L}_j,\dots,L_n)
\end{aligned}$$

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