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Quadratic Involutions on Binary Forms

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1. Introduction

Given a smooth conic C in the projective plane $\mathbb{P}^2$, a point in $\mathbb{P}^2 \setminus C$ will define an involution (i.e., a degree-2 automorphism) on C. Several familiar objects in the invariant theory of binary forms (such as the quartic catalecticant, or the Hermite invariant) can be defined as sets of divisors that are fixed by such an involution. In this paper we study a wide class of such involutions and the corresponding fixed loci.

We begin with an elementary introduction to the subject. The main results are described in Section 2.4 after the required notation is available. Many of the proofs involve elaborate calculations using the graphical or symbolic method, as in [2; 3]. We have omitted them from this paper for the sake of brevity. Instead, the reader is referred to [5], which is the online archival version of this paper, for complete proofs of all the results announced here. We have made an attempt to ensure that such a division of the text does not lead to a breach of continuity.

The reader may consult [9; 11; 16] for classical introductions to the invariant theory of binary forms and [7; 14; 15; 17] for more modern accounts.

1.1. Representations of SL_2

Throughout, the base field will be $\mathbb{C}$ (complex numbers). Let V denote a two-dimensional $\mathbb{C}$-vector space. For a nonnegative integer m, let $S_m = \mathrm{Sym}^m\, V$ denote the mth symmetric power. If $x = \{x_1, x_2\}$ is a basis of V, then S_m can be identified with the space of binary forms of order m in the variables x. The $\{S_m : m \geq 0\}$ are a complete set of finite-dimensional irreducible representations of $\mathrm{SL}(V)$ (see e.g. [13, Sec. I.9]).

Following a notation introduced by Cayley, we will write the binary form $\sum_{i=0}^{d} a_i \binom{d}{i} x_1^{d-i} x_2^i$ as $(a_0, \dots, a_d \between x_1, x_2)^d$.

1.2. Transvectants

Given integers $m, n \geq 0$ and $0 \leq r \leq \min(m,n)$, we have a transvectant morphism (see [4])

$$S_m \otimes S_n \to S_{m+n-2r}.$$

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