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The ∞-Poincaré Inequality in Metric Measure Spaces

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1. Introduction

A useful feature of the Euclidean n-space, $n \geq 2$, is that every pair of points x and y can be joined not only by the line segment $[x, y]$ but also by a large family of curves whose length is comparable to the distance between the points. Once one has found such a "thick" family of curves, the deduction of important Sobolev and Poincaré inequalities is an abstract procedure in which the Euclidean structure no longer plays a role.

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. In this type of inequality, the derivative itself is not needed and only the size of the function's gradient is used; a nice discussion of this can be found in [17]. This is the idea behind generalizations of Poincaré inequalities in spaces where we may not have a linear structure. Heinonen and Koskela [8; 9] introduced a notion of "upper gradients", which serve the role of derivatives in a metric space X. A nonnegative Borel function g on X is said to be an *upper gradient* for an extended real-valued function u on X if $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$ for every rectifiable curve $\gamma\colon [a, b] \to X$. The following Poincaré inequality is now standard in literature on analysis in metric measure spaces.

Definition 1.1. Let $1 \leq p < \infty$. We shall say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that, for every Borel measurable function $u\colon X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g\colon X \to [0, \infty]$ of u, the pair (u, g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_p r \left(\fint_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}$$

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