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The Pompeiu Formula for Slice Hyperholomorphic Functions

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1. Introduction

The fundamental result that makes complex analysis into a new discipline, independent from the theory of real variables, is the Cauchy formula, which allows the representation of any holomorphic function through a reproducing holomorphic kernel. This result is in fact an almost immediate application of the Stokes formula, which, in the more general case, offers an integral representation formula for $\mathcal{C}^1$ functions. This general representation formula is often known as the *Pompeiu formula* and can be stated as follows.

Theorem 1.1 (Pompeiu formula). *Let $U \subset \mathbb{C}$ be a bounded open set such that its boundary consists of a union of a finite number of $\mathcal{C}^1$ Jordan curves. If $f \in \mathcal{C}^1(\bar{U})$, we have*

$$f(\zeta) = \frac{1}{2\pi i}\left(\int_{\partial U} \frac{f(z)}{z-\zeta}\,dz + \int_U \frac{\partial f/\partial \bar{z}}{z-\zeta}\,dz \wedge d\bar{z}\right), \quad \zeta \in U.$$

It is clear that, when f is actually holomorphic on U, the Pompeiu formula becomes the usual Cauchy integral formula.

A classical direction for contemporary research has been the search for the analogue of the theory of complex variables when one considers instead functions from the algebra $\mathbb{H}$ of quaternions into itself, or even functions from the Euclidean space $\mathbb{R}^{n+1}$ into the real Clifford algebra $\mathbb{R}_n$. Although several possible theories have been proposed, it is fair to say that the most successful theory in the case of quaternionic functions is the theory of regular functions in the sense of Cauchy–Fueter, which was originally introduced by Fueter and whose main results are summarized in [8; 15; 16]. In the case of Clifford algebra–valued functions, the most successful theory has been the theory of monogenic functions, for which we refer the readers to [1]. In both cases it is possible to reconstruct analogues of the Cauchy kernel and to prove versions of the Cauchy formula. An analogue of the Pompeiu formula is available as well; see [1, Thm. 9.5].

In the last several years, two new notions of hyperholomorphic functions have been suggested by Gentili and Struppa (in the case of quaternionic-valued functions

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