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Lattice Zariski k-ples of Plane Sextic Curves and Z-Splitting Curves for Double Plane Sextics

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1. Introduction

By virtue of the theory of period mapping, lattice theory has become a strong computational tool in the study of complex $K3$ surfaces. In this paper, we apply this tool to the classification of complex projective plane curves of degree 6 with only simple singularities. In particular, we explain the phenomena of *Zariski pairs* from a lattice-theoretic point of view.

A *simple sextic* is a reduced (possibly reducible) complex projective plane curve of degree 6 with only simple singularities. For a simple sextic $B \subset \mathbb{P}^2$, we denote by μ_B the total Milnor number of B, by $\operatorname{Sing} B$ the singular locus of B, by R_B the ADE-type of the singular points of B, and by $\operatorname{degs} B = [d_1, \dots, d_m]$ the list of degrees $d_i = \deg B_i$ of the irreducible components $B_1, \dots, B_m$ of B.

We have the following equivalence relations among simple sextics.

Definition 1.1. Let B and B' be simple sextics.

(1) We write $B \sim_{\text{eqs}} B'$ if B and B' are contained in the same connected component of an equisingular family of simple sextics.

(2) We say that B and B' are of the *same configuration type,* and write $B \sim_{\text{cfg}} B'$, if there exist tubular neighborhoods $T \subset \mathbb{P}^2$ of B and $T' \subset \mathbb{P}^2$ of B' and a homeomorphism $\varphi\colon (T, B) \xrightarrow{\sim} (T', B')$ such that (a) $\deg \varphi(B_i) = \deg B_i$ holds for each irreducible component B_i of B, (b) φ induces a bijection $\operatorname{Sing} B \xrightarrow{\sim} \operatorname{Sing} B'$, and (c) φ is an analytic isomorphism of plane curve singularities locally around each $P \in \operatorname{Sing} B$. Note that R_B and $\operatorname{degs} B$ are invariants of the configuration type. (See [4, Rem. 3] for a combinatorial definition of $\sim_{\text{cfg}}$.)

(3) We say that B and B' are of the *same embedding type,* and write $B \sim_{\text{emb}} B'$, if there exists a homeomorphism $\psi\colon (\mathbb{P}^2, B) \xrightarrow{\sim} (\mathbb{P}^2, B')$ such that ψ induces a bijection $\operatorname{Sing} B \xrightarrow{\sim} \operatorname{Sing} B'$ and such that, locally around each $P \in \operatorname{Sing} B$, ψ is an analytic isomorphism of plane curve singularities.

It is obvious that

$$B \sim_{\text{eqs}} B' \implies B \sim_{\text{emb}} B' \implies B \sim_{\text{cfg}} B',$$

although the converses do not necessarily hold.

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