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The Rationality of the Moduli Space of Genus-4 Curves Endowed with an Order-3 Subgroup of Their Jacobian

Ingrid Bauer & Alessandro Verra

0. Introduction

Let C be a smooth, irreducible complex projective curve of genus g and let $\eta \in \mathrm{Pic}^0(C)$ be a nontrivial nth root of the trivial bundle $\mathcal{O}_C$. For several different reasons, special attention has been paid, now and in the recent past, to the moduli spaces $\mathcal{R}_{g,n}$ of pairs (C,η) as above and to its possible compactifications (see e.g. [CapCasC]).

For instance, they are generalizations of the case $n = 2$, the so-called *Prym moduli spaces,* usually denoted by $\mathcal{R}_g$. Since they are related to the theory of Prym varieties, the interest in this case occupies a prominent position. In particular, many results on the Kodaira dimension of $\mathcal{R}_g$ are now available, while classical geometric descriptions of $\mathcal{R}_g$ exist for $g \leq 7$. More precisely, let us mention that Farkas and Ludwig [FLu] proved that $\mathcal{R}_g$ is of general type for $g \geq 14$ and $g \neq 15$. On the other hand, unirational parameterizations of $\mathcal{R}_g$ are known for $g \leq 7$ [Cat, D, Do, ILoS, Ve1, Ve2].

One can also consider the moduli spaces $\mathcal{R}_{g,\langle n\rangle}$ of pairs $(C, \mathbb{Z}/n\mathbb{Z})$, where C is a smooth, irreducible complex projective curve of genus g and $\mathbb{Z}/n\mathbb{Z}$ is a cyclic subgroup of order n of $\mathrm{Pic}^0(C)$. As $\mathcal{R}_{g,\langle 2\rangle} = \mathcal{R}_{g,2}$, these mouli spaces are generalizing the Prym moduli spaces in a (slightly) different way. In contrast to the case $n = 2$, not very much is known about $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ for $n > 2$. In particular, the (probably short) list of all pairs (g,n) such that $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ have negative Kodaira dimension is not known.

The rationality of $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ has been proved in some cases of very low genus: the case of $\mathcal{R}_4$ is a result of Catanese [Cat]. The rationality of $\mathcal{R}_3$ was proved by Katsylo in [Ka]. Independent proofs are also due to Catanese and to Dolgachev; see [D] (also for $\mathcal{R}_2$). Recently, the rationality of $\mathcal{R}_{3,3}$ and of $\mathcal{R}_{3,\langle 3\rangle}$ has been proven by Catanese and the first author [BCat].

To complete the picture, we recall that $\mathcal{R}_{1,n}$ is an irreducible curve for every prime n and that its geometric genus is well known.

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