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The Pointwise Convergence of Möbius Maps

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1. Introduction

In 1957, Piranian and Thron [6] classified the possible limits of a pointwise convergent sequence of Möbius maps acting in the extended complex plane. Here we consider the problem for Möbius maps acting in higher dimensions.

Let g_n be any sequence of Möbius maps of the extended complex plane $\mathbb{C}_\infty$ onto itself. Let C be the set of points z at which the sequence $g_n(z)$ converges and, for z in C, let $g(z) = \lim_n g_n(z)$.

Theorem A [6]. *Suppose that $C \neq \emptyset$. Then one of the following possibilities occurs*:

(a) $C = \mathbb{C}_\infty$, *and g is a Möbius map*;
(b) $C = \mathbb{C}_\infty$, *and g is constant on the complement of one point but not on* $\mathbb{C}_\infty$;
(c) $C = \{z_1, z_2\}$ *and* $g(z_1) \neq g(z_2)$; *or*
(d) *g is constant on C.*

It is clear that other possibilities can arise in higher dimensions; for example, the sequence of iterates of a nontrivial rotation in $\mathbb{R}^3$ converges on, and only on, the axis of the rotation and at ∞. Here, we establish the corresponding result in higher dimension, and we replace the arguments about the coefficients of the g_n used in [6] by geometric arguments. We shall see that, even in two dimensions, the two cases in which g takes precisely two values play very different roles in the discussion; in fact, (c) is closer to (a) than to (b). The following similar result for quasiconformal mappings is known [8, pp. 69–73].

Theorem B. *Let D be a subdomain of $\mathbb{R}^{k+1}$, and let f_n be a sequence of K-quasiconformal mappings of D into $\mathbb{R}^{k+1}$ that converges pointwise on D to a function f. Then one of the following possibilities occurs*:

(a) *f is a K-quasiconformal map of D onto some domain D'*;
(b) *f takes precisely two values on D, one of which is taken at one point only*; *or*
(c) *f is constant on D.*

However, Theorem B does not subsume Theorem A, for C need not be a domain; indeed, the problem of characterizing the possible sets C in Theorem A is not

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