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On the Restriction Conjecture

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1. Introduction

The restriction conjecture is a challenging open problem in Fourier analysis. Denoting by

$$\hat{f}(\zeta) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i \langle x, \zeta \rangle} dx$$

the Fourier transform of a C_0^∞ function on $\mathbf{R}^d$ and by $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$ the unit sphere in $\mathbf{R}^d$, the restriction conjecture (RC henceforth) states that, for every $1 \le p < \frac{2d}{d+1}$ and $q \ge \frac{d-1}{d+1} p'$, the following inequality holds:

$$\sup_{F \in C_0^\infty(\mathbf{R}^d)} \frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1}, d\sigma)}}{\|F\|_{L^p(\mathbf{R}^d)}} \le C, \tag{1.1}$$

where $d\sigma(\zeta)$ denotes surface measure on $\mathbf{S}^{d-1}$ and $\mathbf{R}^+ = (0, \infty)$. Here C is a constant that depends only on p, q, and d, and p' is the dual exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The conditions on p and q are optimal (see [10]). The RC has been proved in the case $d = 2$ by Fefferman and Stein (see [6]) and is still open in higher dimensions. When $d > 2$ only partial results are known; one of these results is the Stein–Tomas theorem [9; 13], which asserts that the RC holds whenever $1 \le p < \frac{2(d+1)}{d+3}$ and every $q \ge \frac{d-1}{d+1} p'$. See also [10]. When $p = \frac{2(d+1)}{d+3}$ we have $\frac{d-1}{d+1} p' = 2$, and the exponent $q = 2$ plays a crucial role as it allows a reduction of (1.1) to the equivalent "dual" inequality

$$\left\| \int_{\mathbf{S}^{d-1}} \hat{F}(\zeta) e^{2\pi i \langle x, \zeta \rangle} \, d\sigma(\zeta) \right\|_{L^{p'}(\mathbf{R}^n)} \le C \|F\|_{L^p(\mathbf{R}^n)}$$

via a TT^* technique. The case $q < 2$ cannot be handled with the same technique and requires more delicate work.

When $\frac{2(d+1)}{d+3} < p < \frac{2d}{d+1}$ we can prove that the ratio in (1.1) is uniformly bounded on special subspaces of $L^p(\mathbf{R}^d)$. For example, it is easy to see that (1.1) holds for every $q \le 2$ and every $p \le \frac{2d}{d+1}$ if $L^p(\mathbf{R}^d)$ is replaced by the Sobolev space $W^{s,p_0}(\mathbf{R}^d)$, where $p_0 = \frac{2(d+1)}{d+3}$ and $s \ge \frac{d-1}{d(d+1)}$. By the Sobolev embedding theorem, the latter embeds in $L^p(\mathbf{R}^d)$ for every $p \le \frac{2d}{d+1}$.

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