Michigan Math. J. 51 (2003)

Filtrations, Hyperbolicity, and Dimension for Polynomial Automorphisms of $\mathbb{C}^n$

Rasul Shafikov & Christian Wolf

1. Introduction

Let f be a polynomial automorphism of $\mathbb{C}^n$. We denote by $\hat{f}$ the natural extension of f to a meromorphic map in $\mathbb{P}^n$. Let I^+ denote the indeterminacy set of $\hat{f}$. Analogously, we denote by I^- the indeterminacy set of $\widehat{f^{-1}}$. We say that f is *regular* if f has degree greater than 1 and if $I^+ \cap I^- = \emptyset$.

In the case $n = 2$, the class of regular automorphisms consists of polynomial automorphisms with nontrivial dynamics—that is, finite compositions of generalized Hénon maps (see e.g. [BS1; FM; FS]). In fact, regular polynomial automorphisms can be considered as a natural generalization of complex Hénon maps to higher dimensions. Higher-dimensional regular maps are for instance the so-called shift-like automorphisms studied by Bedford and Pambuccian [BP]. For further examples we refer to Section 2. We point out that, unlike the two-dimensional case, for $n > 2$ there exist polynomial automorphisms with nontrivial dynamics that are not regular (see e.g. [CF]).

The notion of regular polynomial automorphisms was introduced by Sibony [Si], who comprehensively studied these maps using, in particular, methods from pluripotential theory.

In this paper we study the dynamics of regular polynomial automorphisms from a different point of view: We introduce the notion of hyperbolicity for a regular polynomial automorphism f and study its dynamics. That is, we classify the orbits of f analogously to the case of complex Hénon maps in [BS1]. Finally, we study the Hausdorff and box dimension of the Julia sets of f. We derive estimates for these dimensions in the hyperbolic as well as in the nonhyperbolic case.

We will now describe our results in more detail.

Let f be a regular polynomial automorphism of $\mathbb{C}^n$. We define $K^\pm = \{p \in \mathbb{C}^n : \{f^{\pm k}(p) : k \in \mathbb{N}\}$ is bounded$\}$ and the filled-in Julia set by $K = K^+ \cap K^-$. Furthermore, we define the sets $J^\pm = \partial K^\pm$ and $J = J^+ \cap J^-$. The set $J^\pm$ is called the forward/backward Julia set, and J is the Julia set of f (see Section 2 for details).

We construct a filtration of $\mathbb{C}^n$ that has particular escape properties for the orbits of f (see Proposition 3.1). For $n = 2$, the existence of a filtration was already

Received July 22, 2002. Revision received December 11, 2002.

Christian Wolf was partially supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, through FCT's Funding Program.