COMPACT SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In this paper we investigate differential equations of the form

$$(*) \qquad y' + Uy = F(\cdot, y, \mu)$$

in a complex Banach space B. We assume that either $U \in L(B, B)$ and the spectrum of U (denoted by sp U) is in the right half-plane, or else U is a semigroup generator. Our main objective is the study of compact solutions of (*), that is, solutions whose range has a compact closure. This problem seems interesting since it includes periodic and almost-periodic solutions, and since it leads to the approximation of compact solutions to (*) by solutions of equations in finite-dimensional spaces. The continuity of compact solutions with respect to a parameter has been investigated by Taam [4]. We shall also investigate the continuity and analyticity of these solutions as functions of the parameter μ, where μ lies in a complex Banach space X.

The paper is divided into three sections. In Section 1 we study compact solutions of (*) in an arbitrary complex Banach space B. In Section 2, we let B be a Banach space with a basis, and we prove approximation theorems for the compact solutions of (*). In Section 3 we seek compact solutions to (*) for the case where U is a semigroup generator, and then we use the results of Sections 1 and 2 to get approximation theorems for this case.

1. SOLUTIONS IN A COMPLEX BANACH SPACE

Let R denote the real line. The norm of a vector x in B is written as $\|x\|$. For a function f on R into B, we write

$$\|f\|_\infty = \sup \{ \|f(t)\| : t \in R \}.$$

The above is called the *uniform norm* of f.

We say that a function f from R into B is *compact* if f(R) has a compact closure.

The family of functions from R into B that are Bochner integrable on every interval of unit length, and for which

$$\|f\|_s = \sup \left\{ \int_t^{t+1} \|f(s)\| \, ds : t \in R \right\}$$

is finite will be designated by BUL. We call $\| \ \|_s$ the *uniform* L_1-*norm.*

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