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Twins of k-Free Numbers and Their Exponential Sum

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1. Introduction

For any integer $k \geq 2$, let $\mu_k(n)$ denote the characteristic function on the set of k-free numbers; that is, $\mu_k(n) = 0$ if there is a prime p with $p^k \mid n$, and $\mu_k(n) = 1$ otherwise. A twin of k-free numbers is a natural number n such that $\mu_k(n) = \mu_k(n+1) = 1$. It has long been known that the set of these twins has positive density

$$\varrho = \varrho_k = \prod_p \left(1 - \frac{2}{p^k}\right); \tag{1.1}$$

although the first explicit reference to an asymptotic formula for the counting function

$$A_k(x) = \sum_{n \leq x} \mu_k(n)\mu_k(n+1)$$

seems to be a paper by Carlitz [2], the estimate

$$A_k(x) = \varrho x + O(x^{2/(k+1)+\varepsilon}) \tag{1.2}$$

is at least implicit in the work of Evelyn and Linfoot [4] and Estermann [3]. The latter formula (1.2) was then proved in refined form, with x^ε replaced by $(\log x)^{4/3}$, by Mirsky [7]. More recently, Heath-Brown [5] considered the case $k = 2$ and obtained (1.2) with $O(x^{7/11+\varepsilon})$ in place of $O(x^{2/3+\varepsilon})$.

In this paper we study the exponential sum

$$S(\alpha) = S_k(\alpha) = \sum_{n \leq x} \mu_k(n)\mu_k(n+1)e(\alpha n) \tag{1.3}$$

associated with k-free twins. In recent years there has been an increased interest in the L_1-norm of exponential sums over reasonably dense sets of which the k-free twins form an example. Our first theorem adds to the small stock of such sums for which a nontrivial estimate can be obtained.

Theorem 1. *Let $k \geq 2$. Then*

$$\int_0^1 |S_k(\alpha)|\, d\alpha \ll x^{1/(k+1)+\varepsilon}.$$

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