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Toward a Geometric Characterization of Intrinsic Ultracontractivity for Dirichlet Laplacians

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1. Introduction

Let D be a domain in $\mathbb{R}^2$ and let $p_t^D(x, y)$ be the heat kernel of e^{-tH}, where H is minus one half the Dirichlet Laplacian in D. We assume that φ_0, the eigenfunction of H corresponding to the first eigenvalue λ, is positive—a mild assumption which holds for a very large class of domains, including all those considered in this paper. Because

$$p_t^D(x, y) \le \frac{1}{(2t\pi)^{n/2}} \exp\left\{\frac{-|x-y|^2}{2t}\right\},$$

the symmetric Markovian semigroup e^{-tH} is ultracontractive; that is, it maps $L^2(D)$ into $L^\infty(D)$ for all $t > 0$. Following [11], we say that D is *intrinsically ultracontractive* (IU) if the symmetric Markovian semigroup $e^{-t\tilde{H}}$ in $L^2(D, \varphi_0^2\,dx)$ given by the kernel

$$\tilde{p}_t^D(x, y) = \frac{e^{\lambda t} p_t^D(x, y)}{\varphi_0(x)\varphi_0(y)}$$

is ultracontractive. This is equivalent (see [11]) to the existence of $a_t > 0$ and $b_t > 0$ depending only on D and t and such that, for all $x, y \in D$,

$$b_t \varphi_0(x)\varphi_0(y) \le p_t^D(x, y) \le a_t \varphi_0(x)\varphi_0(y). \tag{1}$$

Because of its analytic and probabilistic consequences, intrinsic ultracontractivity has been widely studied by many authors (see e.g. [3; 4]). Sufficient conditions for IU can be found in [2; 8; 11]. The results in these papers do not give necessary and sufficient conditions. It seems to be very difficult (and perhaps impossible) to find a geometric characterization for IU without restricting to some subclasses of domains. However, there is a conjecture for a geometric characterization of IU for a certain class of simply connected domains. We will state it later in this section. The purpose of this paper is to provide some partial results related to this conjecture. Before we state the conjecture and our main results, we need to introduce some notation and present some definitions.

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