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Inflections of Toric Varieties

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Let $V = \{m_0, \dots, m_t\}$ be a set of distinct lattice points in $\mathbb{Z}^n_{\geq 0}$ with $m_0 = \vec{0}$. Associated with V is an affine monomial map

$$\begin{aligned} v\colon \mathbb{C}^n &\to \mathbb{C}^{t+1}, \\ x &\mapsto (1, x^{m_1}, \dots, x^{m_t}), \end{aligned}$$

where x^{m_i} stands for the monomial $x_1^{m_{i1}} x_2^{m_{i2}} \cdots x_n^{m_{in}}$. (The ordering of the lattice points will not be important. The lattice point $m_0 = \vec{0}$ is included, anticipating the move to projective space.) As will be described carefully in Section 1, the span of the derivatives of v up to order k at a point p determines the osculating space of order k at p. If the dimension of this osculating space is smaller than expected then we say that v is inflected at p. In this paper, we show how inflection points are related to the lattice points V and use this information to characterize toric varieties with certain extreme inflectional behavior.

The following two theorems are examples of previous work in which varieties are characterized by their inflectional behavior.

Theorem 0.1 [FKPT]. *Let $t = \binom{n+k}{k} - 1$, and let $X \subset \mathbb{P}^t$ be a smooth projective n-fold whose* kth *osculating space is all of $\mathbb{P}^t$ at all points of X; then X is isomorphic to $\mathbb{P}^n$ embedded via the k-fold Veronese mapping.*

Theorem 0.2 [BPT]. *Let $t \geq 2$, and let $X \subset \mathbb{P}^{2t+1}$ be a smooth projective surface not contained in a hyperplane such that the dimension of its* kth *osculating space is $2k$ at all points of X and for all $k \leq t$. Then X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via all global sections of $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(t)$, so X is a rational normal scroll of degree $2t$.*

These two theorems are proved using sophisticated machinery (in the former case, a result of Mori characterizing projective space as the only variety with ample tangent bundle; in the latter, adjunction theory). However, in all cases, the varieties and embeddings turn out to be toric. As might be expected, if we are willing to restrict our attention to toric mappings then we can establish these theorems by fairly easy combinatorics characterizing polytopes with certain properties.

In Section 1, we show that the dimensions of the osculating spaces of v are given by the Hilbert function of the set of lattice points, V. In Section 2 we discuss

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