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# New Moduli Spaces of Pointed Curves and Pencils of Flat Connections

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*Dedicated to William Fulton on the occasion of his 60th birthday*

## 0. Introduction and Plan of the Paper

One of the remarkable basic results in the theory of the associativity equations (or Frobenius manifolds) is that their formal solutions are the same as cyclic algebras over the homology operad $(H_*(\bar{M}_{0,n+1}))$ of the moduli spaces of $n$-pointed stable curves of genus 0. This connection was discovered by physicists, who observed that the data of both types come from models of topological string theories. Precise mathematical treatment was given in [KM1; KM2; KM3].

In this paper we establish a similar relationship between the pencils of formal flat connections (or solutions to the commutativity equations; see 3.1 and 3.2) and homology of a new series $\bar{L}_n$ of pointed stable curves of genus 0. Whereas $\bar{M}_{0,n+1}$ parameterizes trees of $\mathbf{P}^1$s with pairwise distinct nonsingular marked points, $\bar{L}_n$ parameterizes strings of $\mathbf{P}^1$s and all marked points except for two are allowed to coincide (see the precise definitions in 1.1 and 2.1). Moreover, the union of all the $\bar{L}_n$ forms a semigroup rather than an operad, and the role of operadic algebras is taken over by the representations of the appropriately twisted homology algebra of this union (see precise definitions in 3.3).

This relationship was discovered on a physical level in [L1; L2]. Here we give a mathematical treatment of some of the main issues raised there.

This paper is structured as follows. In Section 1 we introduce the notion of $(A, B)$-pointed curves, whose combinatorial structure generalizes that of strings of projective lines as just described. We then describe a construction of "adjoining a generic black point", which allows us to produce families of such curves and their moduli stacks inductively. This is a simple variation of one of the arguments due to Knudsen [Kn1].

In Section 2 we define and study the spaces $\bar{L}_n$, for which we give two complementary constructions. The first one identifies $\bar{L}_n$ with one of the moduli spaces of pointed curves; the second one exhibits $\bar{L}_n$ as a well-known toric manifold associated with the polytope called "permutohedron" in [K2]. These constructions put $\bar{L}_n$ into two quite different contexts and suggest generalizations in different directions.

As moduli spaces, $\bar{L}_n$ become components of the extended modular operad, which we define and briefly discuss in Section 4. We expect that there exists an

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