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Periodic Orbits and Homoclinic Loops for Surface Homeomorphisms

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0. Introduction

Poincaré invented homoclinic orbits, conjectured their existence in the planar three-body problem, and despaired of understanding their complexity. Research by Birkhoff, Cartwright and Littlewood, and Levinson revealed that near transverse homoclinic points there are robust periodic points. A pinnacle of this line of research, and the basis for much of modern dynamical theory, is Smale's "horseshoe" theorem [22]. For a diffeomorphism f of a manifold of any dimension, it states that every neighborhood of a transverse homoclinic point meets a structurally stable, hyperbolic compact invariant set K on which some iterate f^k is topologically conjugate to the shift map on the Cantor set $2^{\mathbf{Z}}$.

Similar results have been obtained under weakenings of the transversality assumption, including work by Burns and Weiss [6], Churchill and Rod [7], Collins [8], Gavrilov and Šilnikov [13; 14], Guckenheimer and Holmes [15], Mischaikow [18], Mischaikow and Mrozek [19], Newhouse [20], and Rayskin [21].

Among many important consequences is the existence of hyperbolic periodic orbits in K of all periods kn, $n \geq 1$. Note, however, that k is not specified in the horseshoe theorem, and in most cases there is no way to estimate it (but see [19]). Collins [8] has shown that a differentiably transverse homoclinic point implies the existence of periodic points of all sufficiently high minimum periods; estimating such periods, however, requires detailed knowledge of the associated homoclinic tangle.

Although the horseshoe theorem guarantees infinitely many periodic orbits, it is insufficient for the existence of a second fixed point. For example, the toral diffeomorphism induced by the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has only one fixed point, even though transverse homoclinic points are dense.

It turns out that, for diffeomorphisms of the plane, even a nontransverse homoclinic point implies a second fixed point; in fact, there is a block of fixed points having index $+1$. But the proof of this (Hirsch [17]), based on Brouwer's plane translation theorem, gives no indication of the location of such a block.

In this paper we consider a saddle fixed point p for an orientation-preserving homeomorphism f of a surface X (definitions will be given in Section 1). Let p'

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