A General Notion of Shears, and Applications

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1. Introduction

In this paper we introduce a generalization of the notions of shears and overshears to arbitrary complex manifolds. The concept is very simple, but it is useful in the study of complex manifolds having very large automorphism groups. We shall explore some of the consequences of this concept in connection with the density property, which we now recall.

In [V1] we introduced the notion of complex manifolds with the *density property*. Recall that a complex manifold M has the density property if the Lie subalgebra of $\mathcal{X}_{\mathcal{O}}(M)$ generated by the complete vector fields on M is a dense subalgebra. More generally, a Lie subalgebra $\mathfrak{g} \subset \mathcal{X}_{\mathcal{O}}(M)$ is said to have the density property if the complete vector fields in $\mathfrak{g}$ generate a dense subalgebra of $\mathfrak{g}$. (So M has the density property if and only if $\mathcal{X}_{\mathcal{O}}(M)$ has the density property.) Another important case occurs when M has a nonvanishing holomorphic n-form ($n = \dim_{\mathbb{C}} M$), that is, a holomorphic volume element ω. We say that (M, ω) has the *volume density property* if the Lie algebra $\mathcal{X}_{\mathcal{O}}(M, \omega) := \{ X \in \mathcal{X}_{\mathcal{O}}(M) \mid L_X \omega = 0 \}$ has the density property. Andersén [A] proved that $(\mathbb{C}^n, dz_1 \wedge \cdots \wedge dz_n)$ has the volume density property, and then Andersén and Lempert [AL] proved that $\mathbb{C}^n$ has the density property. The author showed that for every complex Lie group G, $(G \times \mathbb{C}, \omega)$ has the volume density property, where ω is the unique (up to constant multiple) left (or right) invariant holomorphic volume element on $G \times \mathbb{C}$, and that if G is a Stein Lie group, then $G \times \mathbb{C}$ has the density property. The author also produced several examples of Lie algebras of vector fields with the density property.

In [V2] we used jets to explore the complex structure of (mostly Stein) complex manifolds with the density property. It was shown, among other things, that Stein manifolds with the density property admit open subsets biholomorphic to $\mathbb{C}^n$ and have interesting properties with respect to their embedded submanifolds. Some of the results were known for $\mathbb{C}^n$ through works of Buzzard, Fornæss, Forstnerič, Globevnik, Rosay, Stensønes, and others.

With the usefulness of the density property already established in the literature, some sort of classification or fine structure theorem is very desirable. Such a result seems at the moment very far off, owing in part to the lack of examples. The main

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